

Gradient estimates for degenerate quasi-linear parabolic equations

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Abstract

For a general class of divergence type quasi-linear degenerate parabolic equations with differentiable structure and lower order coefficients form small with respect to the Laplacian we obtain L^q -estimates for the gradients of solutions, and for the lower order coefficients from Kato-type classes we show that the solutions are Lipschitz continuous with respect to the space variable.

1 Introduction and main results

In this paper we study regularity of local weak solutions to general divergence type quasi-linear degenerate parabolic equations with measurable coefficients and lower order terms. This class of equations has numerous applications and has been attracting attention for several decades (see, e.g. the monographs [6, 14, 28], survey [7] and references therein).

Let Ω be a domain in \mathbb{R}^N , $T > 0$. Set $\Omega_T = \Omega \times (0, T)$. We study solutions to the equation

$$(1.1) \quad u_t - \operatorname{div} \mathbf{A}(x, t, u, \nabla u) = b(x, t, u, \nabla u), \quad (x, t) \in \Omega_T.$$

Throughout the paper we suppose that the function $(\mathbf{A}, b) : \Omega_T \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \times \mathbb{R}$ satisfy the Carathéodory condition, that is $(\mathbf{A}, b)(\cdot, u, z)$ is Lebesgue measurable for all $u \in \mathbb{R}, z \in \mathbb{R}^N$, and $(\mathbf{A}, b)(x, t, \cdot, \cdot)$ is continuous for almost all $(x, t) \in \Omega_T$.

We also assume that the following structure conditions are satisfied:

$$(1.2) \quad \begin{aligned} \mathbf{A}(x, t, u, z)z &\geq c_0|z|^p, \quad z \in \mathbb{R}^n, \\ |\mathbf{A}(x, t, u, z)| &\leq c_1(|z|^{p-1} + 1), \\ |b(x, t, u, z)| &\leq g(x)|z|^{p-1} + f(x)(|u|^{p-1} + 1), \end{aligned}$$

where $p \geq 2$, c_1 and c_2 are positive constants and f and g are nonnegative functions.

Let us remind the reader of the notion of a weak solution to equation (1.1). We say that u is a weak solution to (1.1) if $u \in V(\Omega_T) := L^p_{loc}((0, T); W^{1,p}_{loc}(\Omega)) \cap C((0, T); L^2_{loc}(\Omega))$ and for any interval $[t_1, t_2] \subset (0, T)$ the integral identity

$$(1.3) \quad \int_{\Omega} u \psi dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_{\Omega} \{-u \partial_{\tau} \psi + \mathbf{A}(x, t, u, \nabla u) \nabla \psi\} dx d\tau = \int_{t_1}^{t_2} \int_{\Omega} b(x, t, u, \nabla u) \psi dx d\tau$$

for any $\psi \in W_c^{1,p}(\Omega_T)$.

In [18] local boundedness of weak solutions to (1.1) was obtained under optimal conditions on f and g in terms of membership to the nonlinear Kato classes, which are defined below. The main thrust of the result in [18] is the presence of singular lower order coefficients in the structure conditions with optimal assumptions while not assuming anything in addition on the diffusion part.

In what follows we use the notion of the Wolff potential of a function f (cf. [1]), which is defined by

$$W_{\beta,p}^f(x, R) := \int_0^R \frac{dr}{r} \left(\frac{1}{r^{N-\beta p}} \int_{B_r(x)} f(y) dy \right)^{\frac{1}{p-1}},$$

where here and below $B_r(x) = \{z \in \Omega : |z - x| < r\}$, $p > 1$ and $\beta > 0$. For the case $\beta = 1$ it is customary to drop the first index and write $W_p^f(x, R)$. The corresponding non-linear (local) Kato-type classes $K_{\beta,p}$ are defined by

$$(1.4) \quad K_{\beta,p} := \left\{ f \in L_{loc}^1(\Omega) : \lim_{R \rightarrow 0} \sup_{x \in \Omega'} W_{\beta,p}^f(x, R) = 0 \text{ for all } \Omega' \Subset \Omega \right\}.$$

In case $\beta = 1$ we simply write $K_p = K_{1,p}$. The nonlinear Kato class K_p was introduced in [3]. As one can easily see, for $p = 2$, the class K_p reduces to the standard definition of the Kato class with respect to the Laplacian [24], which is extensively used in the qualitative linear theory of elliptic and parabolic second order PDEs. The class K_p turns out to be almost optimal condition on the lower order coefficients also in case of nonlinear p -Laplacian type elliptic and parabolic PDEs for a number of qualitative properties to hold (see [17, 18] and the references therein). A typical example of a singular function in K_p is

$\frac{\mathbf{1}_{B_{1/2}(0)}}{|x|^p \left(\log \frac{1}{|x|} \right)^\alpha}$ with $\alpha > p - 1$, where here and further on $\mathbf{1}_S$ stands for the characteristic function

of the set S . It was proved in [18] that the condition $f, g^p \in K_p$ implies that $u \in L_{loc}^\infty$. In fact, an inspection of the proof there shows that the conditions of membership of the structure coefficients to the corresponding Kato class can be weakened to the requirement that $\sup_{x \in \Omega'} W_p^{g^p+f}(x, 2R)$ is sufficiently small for any subdomain $\Omega' \Subset \Omega$. More precisely, (cf. [18], [19])

there exists $\nu > 0$ such that, if for every subdomain $\Omega' \Subset \Omega$

$$(1.5) \quad \lim_{R \rightarrow 0} \sup_{x \in \Omega'} W_p^{g^p+f}(x, R) < \nu,$$

then $u \in L_{loc}^\infty(\Omega_T)$.

Throughout the paper we assume that f and g satisfy a condition guaranteeing that $u \in L_{loc}^\infty(\Omega_T)$.

We would like to remark that the condition of smallness of $\sup_{x \in \Omega'} W_p^f(x, 2R)$ cannot be distinguished from the Kato type condition $\lim_{R \rightarrow 0} \sup_{x \in \Omega'} W_p^f(x, R) = 0$ if f has only isolated singularities. For $p = 2$ this was already noticed in [2], where the corresponding example was constructed. We give an extension of this example for the general $p \in [2, N)$ in the Appendix.

In this paper we are interested in the estimates of the gradients of solutions to (1.1) with differentiable structure in the diffusion part. The problem of higher regularity of solutions of quasi-linear equations (and systems) has a long history, which started from $C_{loc}^{1,\alpha}$ results for homogeneous elliptic equations (we refer the reader to the well known monographs [11, 13, 14, 21] for the basic results, historical surveys and references). For a general structure divergence type quasi-linear elliptic equations, the Hölder continuity of the gradients of solutions were obtained by DiBenedetto [4] and Tolksdorf [27]. For the case of quasi-linear parabolic equations gradient estimates under different conditions were studied in [5, 15, 16], see also monographs [6, 28] for basic results and some historic comments. Very recently several interesting results on estimates of the gradients of solutions to quasi-linear elliptic and parabolic equations via nonlinear potentials were obtained in [8, 9]. Most of the results in [8, 9] concern the elliptic equations of p -Laplacian type $-\operatorname{div} \mathbf{A}(x, \nabla u) = \nu$ with a measure in the right hand side. The authors give pointwise estimates of the gradients of solutions via a nonlinear Wolff potential of the measure ν , and as a consequence

obtain a sufficient condition for the boundedness of the gradient. In [8] also parabolic equations were studied, and pointwise estimates of solutions and gradients were obtained, but only for the case $p = 2$. While the results in [8, 9] nicely cover the case of general measures on the right hand side, the situation becomes different when the measure ν is absolutely continuous with respect to the Lebesgue measure with locally square integrable density, i.e. $\nu = f dx$ with $f \in L^2_{loc}$, and the condition on f in [8] turns out to be not optimal, which can be seen on explicit examples. We remark that while this paper was already in preparation, the authors were informed about the new preprint [10], where this situation was studied for the elliptic equations and systems, and with the vector field \mathbf{A} depending on ∇u only. The estimates obtained there are expressed in terms of a new potential which is in fact $W_{\frac{2}{3},3}^{f^2}(x, R)$ and which will appear in our main results as well. Below we make a further comparison of our results with [10].

We study a general situation for equation (1.1), that is we allow for the vector field \mathbf{A} in the diffusion part as well as for the right hand side b to depend on all the arguments. To study higher differentiability it is standard to assume that \mathbf{A} is differentiable in x, u and z and that the following ellipticity and growth conditions hold:

$$\begin{aligned} (1.6) \quad & \langle (\partial_z \mathbf{A})\mu, \mu \rangle \geq c_0 |z|^{p-2} |\mu|^2, \quad \forall \mu, z \in \mathbb{R}^N, \\ (1.7) \quad & |\partial_z \mathbf{A}| \leq c_1 (|z|^{p-2} + 1), \\ (1.8) \quad & |\partial_u \mathbf{A}| \leq g_1(x) |z|^{p-2} + f_1(x), \\ (1.9) \quad & |\partial_x \mathbf{A}| \leq g_2(x) |z|^{p-1} + f_2(x), \end{aligned}$$

where f, f_1, f_2, g, g_1, g_2 are nonnegative functions. Without loss of generality, we do not assume dependence of u in the right hand side of (1.6) - (1.9) since u is locally bounded due to (1.5). In the sequel we refer to f, f_1, f_2, g, g_1, g_2 as to the structure coefficients (cf., e.g. [6, Chap. VIII], see also Remark 1.7 below).

Our aim here is to reveal most general conditions on the structure coefficients guaranteeing higher integrability and boundedness of the gradients of solutions. To formulate our results, we need to introduce some additional classes playing special roles in the results.

The class $K_{\frac{2}{3},3}$ defined in (1.4) with $\beta = \frac{2}{3}$ and $p = 3$, with a typical example of a singular function in $K_{\frac{2}{3},3}$ as $\frac{\mathbf{1}_{B_{1/2}(0)}}{|x|^2 \left(\log \frac{1}{|x|}\right)^\alpha}$ with $\alpha > 2$, already appeared in structure conditions in [17] as \tilde{K}_2 .

We also need to introduce a class of form bounded function with respect to the Laplacian with form bound $\beta > 0$, which we further denote by PK_β .

We say that F is form bounded with respect to the Laplacian with form bound $\beta > 0$ and write $F \in PK_\beta$ if $F \in L^1_{loc}(\Omega)$ and there exists $C \geq 0$ such that for all $\theta \in C_0^\infty(\Omega)$

$$\left| \int_{\Omega} F \theta^2 dx \right| \leq \beta \int_{\Omega} |\nabla \theta|^2 dx + C \int_{\Omega} \theta^2 dx.$$

We will also need the class of infinitesimally form bounded function with respect to the Laplacian, which we further denote by PK_0 , and the class of form bounded function with respect to the Laplacian, which is denoted by PK . These classes are defined by $PK_0 = \bigcap_{\beta > 0} PK_\beta$ and $PK = \bigcup_{\beta > 0} PK_\beta$. All the three

classes became indispensable in many problems in PDE theory. Their complete characterization can be found in [22], [23]. For comparison with the Kato type classes, an example of a singular function in PK is $\frac{\mathbf{1}_{B_{1/2}(0)}}{|x|^2}$, while an example of a member of PK_0 is $\frac{\mathbf{1}_{B_{1/2}(0)}}{|x|^2 \left(\log \frac{1}{|x|}\right)^\alpha}$ with $\alpha > 0$. We also need

local versions of the above classes. Namely, we say that $F \in PK_\beta^{loc}$ (respectively, PK_0^{loc} , PK^{loc}) if $F \mathbf{1}_{\Omega'} \in PK_\beta$ (respectively, PK_0 , PK) for any $\Omega' \Subset \Omega$.

While our main object in this paper is the general equation, it seems worth giving an example of a simpler equation which would illustrate the results below, and which seems to be of independent interest. Let us consider the nonhomogeneous evolution p -Laplace equation $u_t - \Delta_p u = f$. It follows from our results below that if $f^2 \in PK_0$ then the gradient of any weak solution u is in L^q_{loc} for any $q < \infty$, while if $f^2 \in K_{\frac{2}{3},3}$ then $\nabla u \in L^\infty_{loc}$. So, for $f(x) = \frac{1}{|x| \left(\log \frac{1}{|x|}\right)^\alpha} \mathbf{1}_{B_{1/2}(0)}$ with $\alpha > 0$ we have that $\nabla u \in L^q_{loc}$ for any

$q < \infty$, and with $\alpha > 1$, the conclusion is that $\nabla u \in L_{loc}^\infty$, and hence every solution is locally Lipschitz continuous with respect to the spatial variables.

Our strategy is the following. We first show that under some general assumptions on the structure coefficients there exists a local weak solution to (1.1) whose space Hessian exists almost everywhere and the space gradient is in L_{loc}^q for an arbitrary large q , in a cylinder $Q = B_R \times (t_1, t_2) \Subset \Omega_T$ provided the Wolff potentials $\sup_{x \in B_R} W_p^f(x, 2R)$ and $\sup_{x \in B_R} W_p^{g^p}(x, 2R)$ are sufficiently small. This constitutes an existence result. The required a priori estimates are obtained by a finite number of iterations of Moser type. The main assumption here is that all squares of structure coefficients are infinitesimally form bounded with respect to the Laplacian. Next, under some mild additional assumption on f_1 and g_1 , for instance, $f_1, g_1^p \in K_p$, we prove that every weak solution to (1.1) in Ω_T has the same smoothness. In the proof of this result we follow the idea of Tolksdorf [27], comparing the solution u to (1.1) on a small cylinder, with a smooth solution to an auxiliary initial boundary value problem in Q with u as initial boundary value data and the equation satisfying the same structure condition as (1.1). A significant difference between our situation and that in [27] is that we do not rely on a priori Hölder continuity (or even continuity) of the weak solution to (1.1) but rather on the property of smallness of the Wolff potentials $\sup_{x \in B_R} W_p^{g^p}(x, 2R)$ and $\sup_{x \in B_R} W_p^{g_1^p}(x, 2R)$ (see Lemma 1.9). The next step is to obtain the supremum estimates of the gradient. This requires stronger assumptions on the structure coefficients. The technique we use to achieve the result is a parabolic version of the Kilpeläinen–Malý technique [12], [21] (see [18, 26]).

Our first result concerns the existence of weak solutions to (1.1) with integrable powers of the gradient. Further on we distinguish between the gradient $\nabla \xi$ of a scalar function ξ and the spatial derivative $D\zeta$ of a vector valued function ζ . We set $[D\zeta]_{kl} = \partial_{x^l} \zeta_k$. The space $\mathbb{R}^{N \times N}$ of matrices is equipped with the Hilbert-Schmidt norm: for $M = \{m_{kl}\} \in \mathbb{R}^{N \times N}$, we set $|M|^2 \equiv |M|_{HS}^2 = \sum_{kl} m_{kl}^2$.

Theorem 1.1. *Let Q denote the cylinder $B_R \times (t_1, t_2)$ such that $Q \Subset \Omega_T$. Let $v \in V(\Omega_T) \cap L_{loc}^{p'}((0, T); W_{loc}^{-1, p'}(\Omega))$, and let \mathbf{A} and b satisfy the structure conditions (1.2) and (1.6)–(1.9) with $(f^2 + f_1^2 + g_1^2 + f_2^2 + g_2^2)\mathbf{1}_{B_R} \in PK$. Assume that $\sup_{x \in B_R} W_p^{g^p+f}(x, 2R)$ is sufficiently small. Then there exists a solution u to (1.1) in Q satisfying $u = v$ on the parabolic boundary $\mathcal{P}Q$ of Q , such that, for every $l > 0$ and $q \geq p$ and every cylinder $Q' = B' \times (t'_1, t'_2) \Subset Q$ there exist constants β, γ such that*

$$(1.10) \quad \text{ess sup}_{t \in (t'_1, t'_2)} \int_{B'} |\nabla u|^{q-p+2} dx + \iint_{Q'} \left| D(\nabla u(|\nabla u| - l)_+^{\frac{q}{2}-1}) \right|^2 dx d\tau \leq \gamma,$$

provided $(g_1^2 + g_2^2)\mathbf{1}_{B_R} \in PK_\beta$. In particular, if $(g_1^2 + g_2^2)\mathbf{1}_{B_R} \in PK_0$ then $\nabla u \in L_{loc}^q(Q)$ for every $q < \infty$.

Moreover, there exist sequences of Carathéodory functions $(\mathbf{A}_n, b_n) : Q \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \times \mathbb{R}$ and $u_n \in L_{loc}^2((t_1, t_2); W_{loc}^{2,2}(B_R)) \cap C((t_1, t_2); W_{loc}^{1,2}(B_R))$ satisfying $\partial_t u_n - \text{div } A_n(u_n, \nabla u_n) = b_n(u_n, \nabla u_n)$, $u_n = v$ on $\mathcal{P}Q$, such that $(\mathbf{A}_n, b_n)(x, t, s, z) \rightarrow (\mathbf{A}, b)(x, t, s, z)$ as $n \rightarrow \infty$ for a.a. $(x, t) \in Q$ and all $(s, z) \in \mathbb{R} \times \mathbb{R}^N$, that $u_n \rightarrow u$ and $\nabla u_n \rightarrow \nabla u$ as $n \rightarrow \infty$ pointwise a.e. on Q , and that (\mathbf{A}_n, b_n) satisfies the structure conditions (1.2) and (1.6)–(1.9) with the same constants c_0 and c_1 and functions f and g , and smooth functions $f_{1,n}, g_{1,n}, f_{2,n}, g_{2,n}$, replacing f_1, g_1, f_2, g_2 , respectively, satisfying the PK conditions with the same constants, and $f_{i,n} \rightarrow f_i$ and $g_{i,n} \rightarrow g_i$, $i = 1, 2$ pointwise a.e.

The next theorem establishes the same smoothness as above, for all solutions to (1.1).

Theorem 1.2. *Let \mathbf{A} and b satisfy the structure conditions (1.2) and (1.6)–(1.9) with $f^2, f_1^2, f_2^2, g_2^2 \in PK^{loc}$. Let u be a weak solution to (1.1) in Ω_T . There exists $\nu > 0$ such that, if for all $\Omega' \Subset \Omega$,*

$$\lim_{R \rightarrow 0} \sup_{x \in \Omega'} \left[W_p^f(x, R) + W_p^{f_1^{\frac{p}{p-1}}}(x, R) + W_p^{g^p}(x, R) + W_p^{g_1^p}(x, R) \right] < \nu,$$

then, for every $q \geq p$ and $l > 0$, there exists $\beta > 0$ such that

$$\nabla u \in L_{loc}^\infty((0, T); L_{loc}^{q-p+2}(\Omega)) \quad \text{and} \quad \nabla u(|\nabla u| - l)_+^{\frac{q}{2}-1} \in L_{loc}^2((0, T); W_{loc}^{1,2}(\Omega))$$

provided $g_2^2 \in PK_\beta^{loc}$. In particular, if $g_2^2 \in PK_0^{loc}$ then $\nabla u \in L_{loc}^q(Q)$ for every $q < \infty$.

Finally, we give sufficient conditions for the boundedness of the gradient of solutions.

Theorem 1.3. *Let \mathbf{A} and b satisfy structure conditions (1.2) and (1.6)–(1.9). Let u be a weak solution to (1.1). There exists $\nu > 0$ such that, if for all $\Omega' \Subset \Omega$,*

$$\limsup_{R \rightarrow 0} \sup_{x \in \Omega'} \left[W_p^{g^p}(x, R) + W_p^{g_1^p}(x, R) \right] < \nu,$$

and

$$\limsup_{R \rightarrow 0} \sup_{x \in \Omega'} \left[W_{\frac{2}{3},3}^{f^2}(x, R) + W_{\frac{2}{3},3}^{g^2}(x, R) + W_{\frac{2}{3},3}^{f_1^2}(x, R) + W_{\frac{2}{3},3}^{g_1^2}(x, R) + W_{\frac{2}{3},3}^{f_2^2}(x, R) + W_{\frac{2}{3},3}^{g_2^2}(x, R) \right] < \nu,$$

that is, for any $\Omega' \Subset \Omega$,

$$(1.11) \quad \limsup_{R \rightarrow 0} \sup_{x \in \Omega'} \int_0^R \frac{dr}{r} \left(\frac{1}{r^{N-2}} \int_{B_r(x)} (f(y)^2 + g(y)^2 + f_1(y)^2 + g_1(y)^2 + f_2(y)^2 + g_2(y)^2) dy \right)^{\frac{1}{2}} < \nu,$$

then

$$\nabla u \in L_{loc}^\infty(\Omega_T),$$

i.e. all solutions to (1.1) are Lipschitz continuous with respect to the spatial variables.

In particular, if $g^p + g_1^p \in K_p$ and $f^2 + g^2 + f_1^2 + g_1^2 + f_2^2 + g_2^2 \in K_{\frac{2}{3},3}$, then every weak solution to (1.1) is locally Lipschitz.

Due to the scaling properties of equation (1.1) one can eliminate the smallness conditions on the coefficients f, f_1 and f_2 . The next statement though a simple consequence of the preceding theorem, gives a generalization of the above result both in the sense of the structure condition on the right hand side b and on the conditions on the structure coefficients f, f_1 and f_2 .

Corollary 1.4. *Let \mathbf{A} satisfy structure conditions (1.2) and b satisfy the structure condition*

$$|b(x, t, u, z)| \leq g(x)|z|^{p-1} + h(x)|u|^{p-1} + f(x).$$

Let u be a weak solution to (1.1). Assume that for every $\Omega' \Subset \Omega$,

$$\sup_{x \in \Omega'} \int_0^R \frac{dr}{r} \left(\frac{1}{r^{N-2}} \int_{B_r(x)} [f(y)^2 + f_1(y)^2 + f_2(y)^2] dy \right)^{\frac{1}{2}} < \infty.$$

Then there exists $\nu > 0$ such that if for every $\Omega' \Subset \Omega$,

$$\limsup_{R \rightarrow 0} \sup_{x \in \Omega'} \left[W_p^{g^p}(x, R) + W_p^{g_1^p}(x, R) \right] < \nu,$$

and

$$\limsup_{R \rightarrow 0} \sup_{x \in \Omega'} \left[W_{\frac{2}{3},3}^{h^2}(x, R) + W_{\frac{2}{3},3}^{g^2}(x, R) + W_{\frac{2}{3},3}^{g_1^2}(x, R) + W_{\frac{2}{3},3}^{g_2^2}(x, R) \right] < \nu,$$

then u is Lipschitz.

Proof. Let $\lambda > 1$ to be chosen later. Let

$$\tau = \lambda^{p-2}, \quad v(x, \tau) = \lambda^{-1}u(x, t).$$

Then v satisfies the equation

$$v_\tau - \operatorname{div} \tilde{\mathbf{A}}(x, \tau, v, \nabla v) = \tilde{b},$$

where $(\tilde{\mathbf{A}}, \tilde{b})(x, \tau, v, z) = \lambda^{1-p}(\mathbf{A}, b)(x, \lambda^{2-p}\tau, \lambda v, \lambda z)$. For \tilde{b} :

$$|\tilde{b}(x, \tau, v, \nabla v)| = \lambda^{1-p}|b(x, t, u, \nabla u)| \leq g|\nabla v|^{p-1} + h|v|^{p-1} + \lambda^{1-p}f.$$

Analogously one can verify the structure conditions on $\tilde{\mathbf{A}}$:

$$(1.12) \quad \langle (\partial_z \tilde{\mathbf{A}})\mu, \mu \rangle \geq c_0 |z|^{p-2} |\mu|^2, \quad \forall \mu, z \in \mathbb{R}^N,$$

$$(1.13) \quad |\partial_z \tilde{\mathbf{A}}| \leq c_1 (|z|^{p-2} + \lambda^{2-p}),$$

$$(1.14) \quad |\partial_u \tilde{\mathbf{A}}| \leq g_1(x) |z|^{p-2} + \lambda^{2-p} f_1(x),$$

$$(1.15) \quad |\partial_x \tilde{\mathbf{A}}| \leq g_2(x) |z|^{p-1} + \lambda^{1-p} f_2(x),$$

So fix λ large enough so that $\tilde{f} = \lambda^{1-p} f$, $\tilde{f}_1 = \lambda^{2-p} f_1$, $\tilde{f}_2 = \lambda^{1-p} f_2$ satisfy the condition of Theorem 1.3 and the assertion follows. \square

Remark 1.5. To compare Theorem 1.3 and Corollary 1.4 with main results in [10], in which *elliptic* equations and systems are studied, we first note that the results in [10] concern the special case of the vector field \mathbf{A} depends on ∇u only, that is when $f_1 = g_1 = f_2 = g_2 = 0$ in structure conditions (1.8), (1.9). Theorem 1.4 in [10] states only the existence of a locally Lipschitz solution to the equation $-\operatorname{div} \mathbf{A}(\nabla u) = b(x, u, \nabla u)$ subject to the Dirichlet boundary condition with boundary data from $W^{1,p}(\Omega)$ under the assumption of smallness of the Wolff type potential $W_{\frac{2}{3},3}$ of $f^2 + g^2$. The assertion that all the solutions are Lipschitz is proved in Theorem 1.1 in [10], where only the particular case $b(x, u, \nabla u) = f(x)$ is considered. So both results follow from Corollary 1.4 as special cases.

Remark 1.6. As a consequence of Theorem 1.3 one can give sufficient conditions of the local boundedness of the gradient of solutions to (1.1) in terms of the structure coefficients belonging to the Lorentz spaces. This is based on the easily verifiable fact that $f \in L^{N,1} \Rightarrow f^2 \in K_{\frac{2}{3},3}$. We do not dwell upon this further, and refer the reader to [10] for an extensive discussion of this point.

Remark 1.7. In all the above results structure condition (1.7) can be replaced by a more general one $|\partial_z \mathbf{A}| \leq c_1 |z|^{p-2} + h_1(x)$ with the requirement $h^2 \in PK_0$ for Theorems 1.1, 1.2, and $h_1^2 \in K_{\frac{2}{3},3}$ for Theorem 1.3 (compare this with (\mathcal{S}_3) in [6, Chap. VIII]). We did not elaborate this further.

1.1 Auxiliary facts

The following lemma provides an inequality of Hardy-type which is useful in the sequel.

Lemma 1.8. *Let $h \in W_{loc}^{1,p}(\Omega)$, $h > 0$. Suppose that $\Delta_p h \in L_{loc}^1(\Omega)$ and $-\Delta_p h > 0$. Then for any $\theta \in \dot{W}^{1,p}(\Omega)$*

$$(1.16) \quad \int_{\Omega} \frac{(-\Delta_p h)}{h^{p-1}} |\theta|^p dx \leq \int_{\Omega} |\nabla \theta|^p dx.$$

If in addition $h \in L^\infty(\Omega)$ then

$$(1.17) \quad \int_{\Omega} (-\Delta_p h) |\theta|^p dx \leq \|h\|_{\infty}^{p-1} \int_{\Omega} |\nabla \theta|^p dx.$$

Proof. First, by the Young inequality note that $pa^{p-1}b - (p-1)a^p \leq b^p$ for any $a, b > 0$ and $p > 1$. Let $\varepsilon > 0$ and $0 < \theta \in C_c^\infty(\Omega)$. Then it follows that

$$\nabla \left(\frac{\theta^p}{(h + \varepsilon)^{p-1}} \right) |\nabla h|^{p-2} \nabla h \leq p \frac{\theta^{p-1} |\nabla h|^{p-1}}{(h + \varepsilon)^{p-1}} |\nabla \theta| - (p-1) \frac{\theta^p |\nabla h|^p}{(h + \varepsilon)^p} \leq |\nabla \theta|^p.$$

Integrating the above and letting $\varepsilon \rightarrow 0$, we obtain (1.16) for $\theta \in C_0^\infty(\Omega)$. The general case follows by approximation. In case $h \in L^\infty(\Omega)$, it follows from (1.16) that

$$\int_{\Omega} (-\Delta_p h) |\theta|^p dx \leq \int_{\Omega} \frac{\|h\|_{\infty}^{p-1}}{h^{p-1}} (-\Delta_p h) |\theta|^p dx \leq \|h\|_{\infty}^{p-1} \int_{\Omega} |\nabla \theta|^p dx.$$

Hence (1.17) follows. \square

Lemma 1.9. Let $f \geq 0$, $f \in L^1_{loc}$ and u be the weak solution to

$$(1.18) \quad -\Delta_p u = f \quad \text{in } B_R, \quad u|_{\partial B_R} = 0.$$

Then there exists $c > 0$ such that

$$\sup_{B_R} u(x) \leq c \sup_{B_R} W_p^f(x, 2R).$$

Proof. Testing (1.18) by u we obtain

$$(1.19) \quad \int_{B_R} |\nabla u|^p dy \leq \sup_{B_R} u(x) \int_{B_R} f(y) dy.$$

By [12, Theorem 4.8] (see also [21, Theorem 2.125]), for $x_0 \in B_R$,

$$(1.20) \quad u(x_0) \leq c \left(\frac{1}{R^N} \int_{B_R(x_0) \cap B_R} u(y)^p dy \right)^{\frac{1}{p}} + c W_p^f(x_0, 2R).$$

Using the Poincaré inequality, (1.19), the Young inequality and the definition of the Wolff potential we have

$$(1.21) \quad \begin{aligned} \left(\frac{1}{R^N} \int_{B_R} u(y)^p dy \right)^{\frac{1}{p}} &\leq c \left(\frac{1}{R^{N-p}} \int_{B_R} |\nabla u(y)|^p dy \right)^{\frac{1}{p}} \\ &\leq c \left(\sup_{B_R} u(x) \right)^{\frac{1}{p}} \left(\frac{1}{R^{N-p}} \int_{B_R} f(y) dy \right)^{\frac{1}{p}} \leq \frac{1}{2} \sup_{B_R} u(x) + c \left(\frac{1}{R^{N-p}} \int_{B_R(x_0)} f(y) dy \right)^{\frac{1}{p-1}} \\ &\leq \frac{1}{2} \sup_{B_R} u(x) + c W_p^f(x_0, 2R), \quad x_0 \in B_R. \end{aligned}$$

Combining (1.20) and (1.21) and taking supremum over B_R we prove the assertion. \square

As a consequence of Lemma 1.8 and Lemma 1.9 we obtain

Corollary 1.10. Let $\theta \in W_0^{1,p}(B_R)$, $0 \leq f \in L^1_{loc}$. Then there exists $\gamma > 0$ such that

$$(1.22) \quad \int_{B_R} f |\theta|^p dx \leq \gamma \sup_{B_{2R}} W_p^f(x, 2R)^{p-1} \int_{B_R} |\nabla \theta|^p dx.$$

Corollary 1.11. Let $\theta \in W^{1,p}(B_R)$, $0 \leq f \in L^1_{loc}$. Then there exists $\gamma > 0$ such that, for every $\rho > R$,

$$(1.23) \quad \int_{B_R} f |\theta|^p dx \leq \gamma \sup_{B_{2\rho}} W_p^f(x, 2\rho)^{p-1} \left(\int_{B_\rho} |\nabla \theta|^p dx + \frac{1}{(\rho-R)^p} \int_{B_\rho} |\theta|^p dx \right).$$

Proof. Let $\xi \in C_c^1(B_\rho)$ be such that $\xi = 1$ on B_R and $|\nabla \xi| \leq \frac{2}{\rho-R}$. Then, by (1.22),

$$\int_{B_R} f |\theta|^p dx \leq \int_{B_\rho} f |\theta \xi|^p dx \leq \gamma \sup_{B_{2\rho}} W_p^f(x, 2\rho)^{p-1} \int_{B_\rho} |\nabla(\theta \xi)|^p dx.$$

Hence the assertion follows. \square

The following proposition which is easy to verify, shows some useful relations between the classes involved.

Proposition 1.12. Let $p, q > 1$, $\alpha, \beta > 0$. Assume that either $\varkappa > \frac{\beta p}{\alpha q} \vee 1$ or $1 \leq \varkappa = \frac{\beta p}{\alpha q} \leq \frac{p-1}{q-1}$. Then there exists $c > 0$ such that, for all $f > 0$ and $R > 0$,

$$W_{\alpha,q}^{f^{\frac{1}{\varkappa}}}(x, R) \leq c R^{\frac{\varkappa \alpha q - \beta p}{\varkappa(q-1)}} \left(W_{\beta,p}^f(x, 2R) \right)^{\frac{p-1}{\varkappa(q-1)}}.$$

In particular, for $p > 2$, if $\sup_{x \in \Omega} W_{\frac{2}{3},3}^{f^2}(x, R) < \infty$ then $|f|^q \in K_p$ for $q \in [1, 2)$ and if $f^p \in K_p$ then $f^2 \in K_2 \subset PK_0$.

Proof. It suffices to prove the first assertion. First observe that there are constants $C \geq c > 0$ dependent on β, p and N only such that, with $r_k = 2^{-k}R$, $k = 0, 1, 2, \dots$,

$$c \sum_{k=1}^{\infty} \left(\frac{1}{r_k^{N-\beta p}} \int_{B_{r_k}(x)} f(y) dy \right)^{\frac{1}{p-1}} \leq W_{\beta, p}^f(x, R) \leq C \sum_{k=0}^{\infty} \left(\frac{1}{r_k^{N-\beta p}} \int_{B_{r_k}(x)} f(y) dy \right)^{\frac{1}{p-1}}.$$

Next, by the Hölder inequality,

$$\left(\frac{1}{r^{N-\alpha q}} \int_{B_r(x)} f(y)^{\frac{1}{\kappa}} dy \right)^{\frac{1}{q-1}} \leq c \left(\frac{1}{r^{N-\beta p}} \int_{B_r(x)} f(y) dy \right)^{\frac{1}{\kappa(q-1)}} r^{\frac{\kappa \alpha q - \beta p}{\kappa(q-1)}}.$$

If $\kappa(q-1) > p-1$ and $\kappa > \frac{\beta p}{\alpha q}$ then, by the Hölder inequality,

$$W_{\alpha, q}^{f^{\frac{1}{\kappa}}}(x, R) \leq c R^{\frac{\kappa \alpha q - \beta p}{\kappa(q-1)}} \left(W_{\beta, p}^f(x, R) \right)^{\frac{p-1}{\kappa(q-1)}}.$$

If $\kappa(q-1) \leq p-1$ and $\kappa \geq \frac{\beta p}{\alpha q}$ then

$$\begin{aligned} W_{\alpha, q}^{f^{\frac{1}{\kappa}}}(x, R) &\leq c R^{\frac{\kappa \alpha q - \beta p}{\kappa(q-1)}} \sum_{k=0}^{\infty} \left(\frac{1}{r_k^{N-\beta p}} \int_{B_{r_k}(x)} f(y) dy \right)^{\frac{1}{p-1}} \sup_k \left(\frac{1}{r_k^{N-\beta p}} \int_{B_{r_k}(x)} f(y) dy \right)^{\frac{1}{\kappa(q-1)} - \frac{1}{p-1}} \\ &\leq c R^{\frac{\kappa \alpha q - \beta p}{\kappa(q-1)}} \left(W_{\beta, p}^f(x, 2R) \right)^{\frac{p-1}{\kappa(q-1)}}. \end{aligned}$$

The inclusion $K_2 \subset PK_0$ is well known in the standard theory of Kato classes [24]. \square

2 Proof of Theorem 1.1

We start with constructing an appropriate local approximation of equation (1.1) and obtaining a priori estimates.

Approximation. For $\varepsilon > 0$ let j_ε be the standard mollifier in \mathbb{R}^N . Denote $\mathbf{A}_\varepsilon = \mathbf{A} * j_\varepsilon + \varepsilon z$, smoothing with respect to x variable only. For b we introduce $b_\varepsilon = b \wedge \frac{1}{\varepsilon} \vee (-\frac{1}{\varepsilon})$. Also set $f_{1, \varepsilon} = f_1 * j_\varepsilon$, $f_{2, \varepsilon} = f_2 * j_\varepsilon$, $g_{1, \varepsilon} = g_1 * j_\varepsilon$ and $g_{2, \varepsilon} = g_2 * j_\varepsilon$. Note that the structure conditions (1.2) and (1.6)–(1.9) hold with $\mathbf{A}_\varepsilon, b_\varepsilon, f_{1, \varepsilon}, f_{2, \varepsilon}, g_{1, \varepsilon}$ and $g_{2, \varepsilon}$ replacing $\mathbf{A}, b, f_1, f_2, g_1$ and g_2 , respectively. Note also that, if $F \in PK_0$ then $F * j_\varepsilon \in PK_0$ for all $\varepsilon > 0$, with the same function $C(\beta)$.

Let Q denote a cylinder $B_R \times (t_1, t_2)$ such that $Q \Subset \Omega_T$. Consider the following approximating equation.

$$(2.1) \quad u_t - \operatorname{div} \mathbf{A}_\varepsilon(x, t, u, \nabla u) = b_\varepsilon(x, t, u, \nabla u), \quad (x, t) \in Q.$$

In the rest of this subsection we study solutions to (2.1) in Q . Our task in the sequel is to obtain estimates which are uniform in ε and which will allow us to pass to the limit $\varepsilon \rightarrow 0$.

In order to simplify the notation, in the rest of this subsection in all proofs we drop subindex ε . We often use the Steklov averaging T_h , $h > 0$, defined by

$$(T_h v)(x, t) = \frac{1}{2h} \int_{-h}^h v(x, t + s) ds.$$

We write $v_h = T_h v$.

Proposition 2.1. *Let $u_\varepsilon \in V(Q)$ be a solution to (2.1) in Q . Then, for every $Q' = B' \times (t'_1, t'_2) \Subset Q'' = B'' \times (t''_1, t''_2) \Subset Q$, there exists $\gamma > 0$ independent of ε such that*

$$\|\partial_t u_\varepsilon\|_{L^{p'}((t'_1, t'_2); W^{-1, p'}(B'))} \leq \gamma \left(\|\nabla u_\varepsilon\|_{L^p(Q'')} + \|u_\varepsilon\|_{L^p(Q'')} + \|f\|_{L^1(Q')}^{\frac{1}{p'}} \right).$$

Proof. To prove the assertion it suffices to show that, for all $\xi \in C_c^1(Q')$,

$$\left| \iint_{Q'} u \partial_t \xi dx dt \right| \leq \gamma \|\nabla \xi\|_{L^p(Q')}.$$

It follows from (2.1) and structure conditions (1.2) that

$$\begin{aligned} \left| \iint_{Q'} u \partial_t \xi dx dt \right| &\leq \iint_{Q'} [|\mathbf{A}(u, \nabla u)| |\nabla \xi| + |b(u, \nabla u)| |\xi|] dx dt \\ &\leq \gamma \|\nabla u\|_{L^p(Q')} (\|\nabla \xi\|_{L^p(Q')} + \|g\xi\|_{L^p(Q')}) \\ &\quad + \gamma \|\nabla \xi\|_{L^1(Q')} + \gamma \|f^{\frac{1}{p}} u\|_{L^p(Q')}^{p-1} \|f^{\frac{1}{p}} \xi\|_{L^p(Q')} + \gamma \|f\|_{L^1(Q')}^{\frac{1}{p'}} \|f^{\frac{1}{p}} \xi\|_{L^p(Q')}. \end{aligned}$$

Corollary 1.10 implies that

$$\|f^{\frac{1}{p}} \xi\|_{L^p(Q')} + \|g\xi\|_{L^p(Q')} \leq \gamma \|\nabla \xi\|_{L^p(Q')}.$$

Finally, from Corollary 1.11 we conclude that

$$\|f^{\frac{1}{p}} u\|_{L^p(Q')}^{p-1} \leq \gamma (\|\nabla u\|_{L^p(Q'')} + \|u\|_{L^p(Q'')}).$$

Hence the assertion follows. \square

Proposition 2.2. *Let $u_\varepsilon \in V(Q)$ be a solution to (2.1) in Q . Then $u_\varepsilon \in L_{loc}^2((t_1, t_2); W_{loc}^{2,2}(Q))$ and $|\nabla u_\varepsilon|^q \nabla u_\varepsilon \in L_{loc}^\infty((t_1, t_2); L_{loc}^2(Q)) \cap L_{loc}^2((t_1, t_2); W_{loc}^{1,2}(Q))$ for all $q \geq \frac{p}{2}$.*

Proof. The assertion follows by the direct approach via finite differences (see, e.g. [6, Section VIII.3] and [13, Section IV.5]). \square

The main result of this subsection is the following a priori estimate.

Proposition 2.3. *Let $Q' = B' \times (t'_1, t'_2) \Subset Q'' = B'' \times (t''_1, t''_2) \Subset Q = B \times (t_1, t_2)$ be cylinders compactly embedded in Ω_T and let $u_\varepsilon \in V(Q)$ be a solution to (2.1) in Q . Assume that \mathbf{A} and b satisfy the structure conditions (1.2) and (1.6)–(1.9) with the functions $(f^2 + f_1^2 + g_1^2 + f_2^2 + g_2^2)\mathbf{1}_{B_R} \in PK$ and that there exists M independent of ε such that $|u_\varepsilon| \leq M$ on Q'' . Then, for every $l > 0$ and $\alpha \geq 0$, there exist constants β and γ independent of ε , such that, if $(g^2 + g_1^2 + g_2^2)\mathbf{1}_{B_R} \in PK_\beta$ then*

$$\begin{aligned} (2.2) \quad &\text{ess sup}_{t \in [t'_1, t'_2]} \int_{B'} |\nabla u_\varepsilon|^{2+2\alpha} dx + \iint_{Q'} \left| D(\nabla u_\varepsilon (|\nabla u_\varepsilon| - l)_+^{\alpha + \frac{p}{2} - 1}) \right|^2 dx d\tau \\ &\leq \gamma \left(\iint_{Q''} (|\nabla u_\varepsilon|^p + F^2 + 1) dx d\tau \right)^{\alpha+1} + \gamma \left(\iint_{Q''} (F^2 + 1) dx d\tau \right)^{\frac{N}{N+2}}, \end{aligned}$$

with $F = f + f_1 + g + g_1 + f_2 + g_2$.

The proof of this proposition is divided into several lemmas, some of which will be used in further argument as well.

Since u_ε is twice weakly differentiable, we can differentiate equation (2.1). This is done in the next lemma.

Lemma 2.4. *Let Q be as in Proposition 2.3 and $u_\varepsilon \in V(Q)$ be a weak solution to (2.1) in Q . Then for every $\zeta \in H_c^1(Q \rightarrow \mathbb{R}^N)$ and for all $t_1 < t'_1 < t'_2 < t_2$, one has*

$$\begin{aligned} (2.3) \quad &\int_{B_R} \langle \nabla u_\varepsilon, \zeta \rangle dx \Big|_{t'_1}^{t'_2} - \int_{t'_1}^{t'_2} \int_{B_R} \langle \nabla u_\varepsilon, \partial_t \zeta \rangle dx dt + \int_{t'_1}^{t'_2} \int_{B_R} \text{tr}\{(D\zeta)(\partial_z \mathbf{A}_\varepsilon) D^2 u_\varepsilon\} dx dt \\ &= - \int_{t'_1}^{t'_2} \int_{B_R} [\text{tr}\{(D\zeta) \partial_x \mathbf{A}_\varepsilon\} + \langle (D\zeta)(\partial_u \mathbf{A}_\varepsilon), \nabla u_\varepsilon \rangle + b_\varepsilon \text{div} \zeta] dx dt. \end{aligned}$$

Proof. First, let $\zeta \in C_c^2(Q \rightarrow \mathbb{R}^N)$. Test equation (2.1) by $-\operatorname{div} \zeta$. Integrating by parts we obtain

$$\int_{B_R} \langle \nabla u_\varepsilon, \zeta \rangle dx \Big|_{t'_1}^{t'_2} - \int_{t'_1}^{t'_2} \int_{B_R} \langle \nabla u, \partial_t \zeta \rangle dx dt + \int_{t'_1}^{t'_2} \int_{B_R} \operatorname{tr}\{(D\zeta)(D\mathbf{A})\} dx dt = - \int_{t'_1}^{t'_2} \int_{B_R} b \operatorname{div} \zeta dx dt.$$

Observing that

$$D\mathbf{A} = (\partial_z \mathbf{A}) D^2 u + \partial_u \mathbf{A} \otimes \nabla u + \partial_x \mathbf{A}$$

we arrive at (2.3). The general case follows by approximation. \square

Remark 2.5. Note that (1.6) implies that, for $M \in \mathbb{R}^{N \times N}$, one has

$$\operatorname{tr}\{M^T (\partial_z \mathbf{A}) M\} \geq c_0 |z|^{p-2} |M|^2.$$

Indeed, let $M = \{m_{kl}\}$. Then

$$\begin{aligned} \operatorname{tr}\{M^T (\partial_z \mathbf{A}) M\} &= \sum_{jkl} m_{kj} (\partial_{z_l} \mathbf{A}_k) m_{lj} = \sum_j \sum_{kl} (\partial_{z_l} \mathbf{A}_k) m_{lj} m_{kj} \\ &\geq \sum_j c_0 |z|^{p-2} \sum_k m_{kj}^2 = c_0 |z|^{p-2} |M|^2. \end{aligned}$$

The proof of Proposition 2.3 is performed by a Moser-type iteration with finite number of steps. The following two lemmas contain the main technical part of the proof.

Lemma 2.6. *Let Q' , Q and u_ε be as in Proposition 2.3. Let $\xi \geq 0$, $\xi \in C^\infty(Q')$ vanishing on the parabolic boundary $\mathcal{P}Q'$. Let $\Phi \in C_b^{0,1}(\mathbb{R})$, $\Phi(0) = 0$ and $\mathcal{G}(s) := \int_0^s \tau \Phi(\tau) d\tau$. Let $\zeta := \nabla u_\varepsilon \Phi(|\nabla u_\varepsilon|) \xi$. Then for almost all (a.a.) $\tau \in (t'_1, t'_2)$ one has*

$$\begin{aligned} &\int_{B'} \mathcal{G}(|\nabla u_\varepsilon(\tau)|) \xi(\tau) dx + \int_{t_1}^\tau \int_{B'} \operatorname{tr}\{(D\zeta)(\partial_z \mathbf{A}_\varepsilon) D^2 u_\varepsilon\} dx dt \\ &\leq \int_{t_1}^\tau \int_{B'} \mathcal{G}(|\nabla u_\varepsilon|) \partial_t \xi dx dt - \int_{t_1}^\tau \int_B [\operatorname{tr}\{(D\zeta) \partial_x \mathbf{A}_\varepsilon\} + \langle (D\zeta)(\partial_u \mathbf{A}_\varepsilon), \nabla u_\varepsilon \rangle + b_\varepsilon \operatorname{div} \zeta] dx dt. \end{aligned}$$

Proof. As before, we write $u_h = T_h u$, where T_h is the Steklov averaging with $h < \min\{t_2 - t'_2, t'_1 - t_1\}$. With notation above set $\zeta_h = \nabla u_h \Phi(|\nabla u_h|) \xi$. We apply $T_h \zeta$ as the test vector function in (2.3):

$$\begin{aligned} &\int_{B_R} \langle \nabla u_h(\tau), \zeta_h(\tau) \rangle dx - \int_{t_1}^\tau \int_{B_R} \langle \nabla u_h, \partial_t \zeta_h \rangle dx dt + \int_{t_1}^\tau \int_{B_R} \operatorname{tr}\{(D\zeta_h) T_h [(\partial_z \mathbf{A}) D^2 u]\} dx dt \\ &= - \int_{t_1}^\tau \int_{B_R} [\operatorname{tr}\{(D\zeta_h) T_h \partial_x \mathbf{A}\} + \langle (DT_h \zeta_h)(\partial_u \mathbf{A}), \nabla u \rangle + T_h[b] \operatorname{div} \zeta_h] dx dt. \end{aligned}$$

Now we pass to the limit as $h \rightarrow 0$. For the first two terms in the left hand side we have

$$\begin{aligned} &\int_{B_R} \langle \nabla u_h(\tau), \zeta_h(\tau) \rangle dx - \int_{t_1}^\tau \int_{B_R} \langle \nabla u_h, \partial_t \zeta_h \rangle dx dt = \int_{t_1}^\tau \int_{B_R} \langle \partial_t \nabla u_h, \nabla u_h \rangle \Phi(|\nabla u_h|) \xi dx dt \\ &= \frac{1}{2} \int_{t_1}^\tau \int_{B_R} (\partial_t |\nabla u_h|^2) \Phi(|\nabla u_h|) \xi dx dt = \int_{t_1}^\tau \int_{B_R} (\partial_t \mathcal{G}(|\nabla u_h|)) \xi dx dt \\ &= \int_{B_R} \mathcal{G}(|\nabla u_h(\tau)|) \xi(\tau) dx - \int_{t_1}^\tau \int_{B_R} \mathcal{G}(|\nabla u_h|) \partial_t \xi dx dt. \end{aligned}$$

Since $\nabla u_h \rightarrow \nabla u$ a.e. as $h \rightarrow 0$, we obtain

$$\begin{aligned} \lim_{h \rightarrow 0} \int_{t_1}^{\tau} \int_{B_R} \mathcal{G}(|\nabla u_h|) \partial_t \xi dx dt &= \int_{t_1}^{\tau} \int_{B_R} \mathcal{G}(|\nabla u|) \partial_t \xi dx dt, \\ \liminf_{h \rightarrow 0} \int_{B_R} \mathcal{G}(|\nabla u_h|(\tau)) \xi(\tau) dx &\geq \int_{B_R} \mathcal{G}(|\nabla u|(\tau)) \xi(\tau) dx, \end{aligned}$$

and the assertion follows. \square

Lemma 2.7. *Let $Q' \Subset Q'' \Subset Q$, u_ε and M be as in Proposition 2.3. For $l > 0$ and $\alpha \geq 0$, let*

$$\Phi_\alpha(s) = (s - l)_+^{2+2\alpha} s^{-2}, \quad \alpha \geq 0, \quad \mathcal{G}_\alpha(s) = \int_0^s r \Phi_\alpha(r) dr.$$

Assume that $(f^2 + g^2 + f_1^2 + g_1^2 + f_2^2 + g_2^2) \mathbf{1}_{B_R} \in PK$. Then, for every $\alpha \geq 0$ and $l > 0$ there exist β and γ independent of ε such that

$$\begin{aligned} (2.4) \quad & \operatorname{ess\,sup}_t \int_{B'} \mathcal{G}_\alpha(|\nabla u_\varepsilon(t)|) dx + \iint_{Q'} |\nabla u_\varepsilon|^{p-2} |D^2 u_\varepsilon|^2 \Phi_\alpha(|\nabla u_\varepsilon|) dx dt \\ & + \iint_{Q''} |\nabla u_\varepsilon|^{p-1} |\nabla |\nabla u_\varepsilon||^2 \Phi'_\alpha(|\nabla u_\varepsilon|) dx dt \\ & \leq \gamma \iint_{Q''} (f^2 + g^2 + f_1^2 + g_1^2 + f_2^2 + g_2^2) dx dt + \gamma \iint_{Q''} |\nabla u_\varepsilon|^p \Phi_\alpha(|\nabla u_\varepsilon|) dx dt \end{aligned}$$

provided $(g^2 + g_1^2 + g_2^2) \mathbf{1}_{B_R} \in PK_\beta$.

Proof. Since $|u| \leq M$ on Q'' , it follows that $|b(u, \nabla u)| \leq g |\nabla u|^{p-1} + \gamma f$ on Q'' .

In the rest of the proof we omit the subscript α in Φ_α and \mathcal{G}_α . Let ξ be the standard cut-off function vanishing on the parabolic boundary of Q'' , which is equal to 1 on Q' .

By Lemma 2.6 with $\zeta = \Phi(|\nabla u|) \xi^2 \nabla u$ as a test function we have

$$\begin{aligned} (2.5) \quad & \sup_t \int_{B''} \mathcal{G}(|\nabla u|(t)) \xi^2 dx + \iint_{Q''} \operatorname{tr}\{(D\zeta)(\partial_z \mathbf{A}) D^2 u\} dx dt \\ & \leq 2 \iint_{Q''} \mathcal{G}(|\nabla u|) \xi \partial_t \xi dx dt + \iint_{Q''} (|\partial_x \mathbf{A}| + |\partial_u \mathbf{A}| |\nabla u| + |b|) |D\zeta| dx dt. \end{aligned}$$

Note that

$$D\zeta = \Phi(|\nabla u|) \xi^2 D^2 u + \Phi'(|\nabla u|) \xi^2 \nabla u \otimes \nabla |\nabla u| + 2\Phi(|\nabla u|) \xi \nabla u \otimes \nabla \xi.$$

Now we estimate the left hand side of (2.5) from below using (1.6)–(1.7), Remark 2.5, the identities $D^2 u \nabla u = |\nabla u| \nabla |\nabla u|$ and $s\Phi'(s) = 2\alpha\Phi(s) + (2+2\alpha)l(s-l)^{1+2\alpha} s^{-2}$:

$$\begin{aligned} \operatorname{tr}\{(D\zeta)(\partial_z \mathbf{A}) D^2 u\} &\geq c_0 \Phi(|\nabla u|) |\nabla u|^{p-2} |D^2 u|^2 \xi^2 + c_0 \Phi'(|\nabla u|) |\nabla u|^{p-1} |\nabla |\nabla u||^2 \xi^2 \\ &\quad - 2(1+l^{2-p}) c_1 \Phi(|\nabla u|) |\nabla u|^{p-1} |\nabla |\nabla u|| \xi |\nabla \xi| \\ &\geq c_0 \Phi(|\nabla u|) |\nabla u|^{p-2} |D^2 u|^2 \xi^2 + \frac{1}{2} c_0 \Phi'(|\nabla u|) |\nabla u|^{p-1} |\nabla |\nabla u||^2 \xi^2 \\ &\quad - c_{l,p,\alpha} \Phi(|\nabla u|) |\nabla u|^p |\nabla \xi|^2. \end{aligned}$$

The first term on the right hand side of (2.5) is estimated using the elementary inequality $\mathcal{G}(s) \leq c_\alpha l^{2-p} \Phi(s)$. The second term is estimated by the Schwartz inequality using the PK_0 condition. To shorten the exposition, we denote $F_\varepsilon := f + f_{1,\varepsilon} + f_{2,\varepsilon}$, $G_\varepsilon := g + g_{1,\varepsilon} + g_{2,\varepsilon}$. Observe that F_ε^2 and G_ε^2 belong to PK with the same constants as F^2 and G^2 , respectively.

It follows from (1.2) and (1.8)–(1.9) that

$$|\partial_x \mathbf{A}| + |\partial_u \mathbf{A}| |\nabla u| + |b| \leq |\nabla u|^{p-1} G_\varepsilon + |\nabla u| f_{1,\varepsilon} + f + f_2.$$

To estimate the right hand side of (2.5) we use the Schwartz inequality and the estimate $\Phi(s)s^p + \Phi'(s)s^{p+1} \leq c_l(\Phi(s)(s-l)_+^p + 1)$ in order to conclude that, for all $\delta > 0$,

$$\begin{aligned}\Phi(|\nabla u|)\xi^2|D^2u||\nabla u|^{p-1}G_\varepsilon &\leq \delta\Phi(|\nabla u|)|\nabla u|^{p-2}|D^2u|^2\xi^2 + \frac{c_l}{\delta}(\Phi(|\nabla u|)(|\nabla u|-l)_+^p + 1)\xi^2G_\varepsilon^2; \\ \Phi'(|\nabla u|)\xi^2|\nabla u|^p|\nabla|\nabla u||G_\varepsilon &\leq \delta\Phi'(|\nabla u|)|\nabla u|^{p-1}|\nabla|\nabla u||^2\xi^2 + \frac{c_l}{\delta}(\Phi(|\nabla u|)(|\nabla u|-l)_+^p + 1)\xi^2G_\varepsilon^2; \\ \Phi(|\nabla u|)\xi|\nabla u|^p|\nabla\xi|G_\varepsilon &\leq \frac{1}{2}(\Phi(|\nabla u|)(|\nabla u|-l)_+^p + 1)\xi^2G_\varepsilon^2 + \frac{1}{2}\Phi(|\nabla u|)|\nabla u|^p|\nabla\xi|^2.\end{aligned}$$

Similarly, since $p > 2$, for every $\sigma > 0$ there exists $c_{l,\alpha,\sigma} > 0$ such that

$$(\Phi(s)s^{2-p} + \Phi'(s)s^{3-p})(1+s^2) \leq \sigma\Phi(s)(s-l)_+^p + c_{l,\alpha,\sigma}.$$

Hence we conclude that, for all $\delta > 0$, there exist $c_{l,\alpha,\delta} > 0$ such that

$$\begin{aligned}\Phi(|\nabla u|)\xi^2|D^2u|(|\nabla u|f_{1,\varepsilon} + f + f_{2,\varepsilon}) &\leq \delta\Phi(|\nabla u|)|\nabla u|^{p-2}|D^2u|^2\xi^2 + \frac{c_l}{\delta}\Phi(|\nabla u|)|\nabla u|^{2-p}(|\nabla u|f_{1,\varepsilon} + f + f_{2,\varepsilon})^2\xi^2 \\ &\leq \delta\Phi(|\nabla u|)|\nabla u|^{p-2}|D^2u|^2\xi^2 + \delta\Phi(|\nabla u|)(|\nabla u|-l)_+^p\xi^2F_\varepsilon^2 + c_{l,\alpha,\delta}\xi^2F_\varepsilon^2; \\ \Phi'(|\nabla u|)\xi^2|\nabla u||\nabla|\nabla u||(|\nabla u|f_{1,\varepsilon} + f + f_{2,\varepsilon}) &\leq \delta\Phi'(|\nabla u|)|\nabla u|^{p-1}|\nabla|\nabla u||^2\xi^2 + \frac{c_l}{\delta}\Phi'(|\nabla u|)|\nabla u|^{3-p}(|\nabla u|f_{1,\varepsilon} + f + f_{2,\varepsilon})^2\xi^2 \\ &\leq \delta\Phi'(|\nabla u|)|\nabla u|^{p-1}|\nabla|\nabla u||^2\xi^2 + \delta\Phi(|\nabla u|)(|\nabla u|-l)_+^p\xi^2F_\varepsilon^2 + c_{l,\alpha,\delta}\xi^2F_\varepsilon^2; \\ \Phi(|\nabla u|)\xi|\nabla u||\nabla\xi|(|\nabla u|f_{1,\varepsilon} + f + f_{2,\varepsilon}) &\leq \frac{1}{2}\Phi(|\nabla u|)|\nabla u|^p|\nabla\xi|^2 + \frac{1}{2}\Phi(|\nabla u|)|\nabla u|^{2-p}(|\nabla u|f_{1,\varepsilon} + f + f_{2,\varepsilon})^2\xi^2 \\ &\leq \frac{1}{2}\Phi(|\nabla u|)|\nabla u|^p|\nabla\xi|^2 + \delta\Phi(|\nabla u|)(|\nabla u|-l)_+^p\xi^2F_\varepsilon^2 + c_{l,\alpha,\delta}\xi^2F_\varepsilon^2.\end{aligned}$$

To complete the proof it remains to estimate the term $\iint_{Q''} \Phi(|\nabla u|)(|\nabla u|-l)_+^p\xi^2(F_\varepsilon^2 + G_\varepsilon^2)dx dt$, which is done by the direct use of the PK condition noting the inequality $|\nabla(\sqrt{\Phi(|\nabla u|)}(|\nabla u|-l)_+^{\frac{p}{2}})|^2 \leq c\Phi(|\nabla u|)(|\nabla u|-l)_+^{p-2}|\nabla|\nabla u||^2$. We omit further details. \square

Proof of Proposition 2.3. To prove the proposition it suffices to show that, for $\alpha > 0$ and a cylinder Q_1 such that $Q' \Subset Q_1 \Subset Q''$,

$$(2.6) \quad \iint_{Q_1} |\nabla u|^{p+2\alpha} dx dt \leq \gamma_\alpha \left(\iint_{Q''} (|\nabla u|^p + F^2 + 1) dx dt \right)^{\alpha+1} + \gamma_\alpha \left(\iint_{Q''} (F^2 + 1) dx dt \right)^{N/(N+2)}.$$

Then the assertion follows from Lemma 2.7.

The proof of (2.6) follows the line of the argument from [6, Ch.VIII, Lemma 4.1]. We will iterate with respect to α as it is done in [6, p.232–233] (with β in place of our 2α). Let $Q^\dagger = (t_1^\dagger, t_2^\dagger) \times B^\dagger$, $Q^\ddagger = (t_1^\ddagger, t_2^\ddagger) \times B^\ddagger$ be such that $Q^\dagger \Subset Q^\ddagger \Subset Q$. Fix $\alpha > 0$. Let Φ_α and \mathcal{G}_α be as in Lemma 2.7 with $l = 1$, $\Psi_\alpha(s) = \int_1^s r^{p/2-1} \sqrt{\Phi_\alpha(r)} dr$ with . Note that $\Psi_\alpha(s) \leq s^{p/2+\alpha}$ and $|\Psi'_\alpha(s)|^2 = s^{p-2}\Phi_\alpha(s)$. Using the definitions of Φ_α , Ψ_α and \mathcal{G}_α and the Sobolev inequality we obtain

$$\begin{aligned}\iint_{Q^\dagger} |\nabla u|^{p+\frac{4}{N}+2\alpha(1+\frac{2}{N})} dx dt &\leq 2^{p+\frac{4}{N}+2\alpha(1+\frac{2}{N})}|Q^\dagger| + \iint_{Q^\dagger \cap \{|\nabla u|>1\}} \Psi_\alpha^2(|\nabla u|)\mathcal{G}_\alpha^{2/N}(|\nabla u|) dx dt \\ &\leq \gamma|Q^\dagger| + \gamma \iint_{Q^\dagger} (|\nabla \Psi_\alpha(|\nabla u|)|^2 + \Psi_\alpha^2(|\nabla u|)) dx dt \left(\sup_t \int_{B^\dagger} \mathcal{G}_\alpha(|\nabla u|)\xi^2 dx \right)^{2/N}.\end{aligned}$$

By Lemma 2.7 we estimate the right hand side of the above inequality, which gives

$$(2.7) \quad \iint_{Q^\dagger} |\nabla u|^{p+\frac{4}{N}+2\alpha(1+\frac{2}{N})} dx dt \leq \gamma|Q^\dagger| + \gamma \left(\iint_{Q^\dagger} (F^2 + |\nabla u|^{p+2\alpha}) dx dt \right)^{1+2/N}.$$

Consider the exhaustion of Q'' by cylinders $Q_0 = Q' \Subset Q_1 \Subset Q_2 \Subset \dots \Subset Q_n \Subset \dots \Subset Q''$. By iterating (2.7) with $\varkappa_n := (1 + \frac{2}{N})^{n-1}$, we obtain

$$(2.8) \quad \iint_{Q_1} |\nabla u|^{p+2\varkappa_n-2} dx dt \leq \gamma_n \left(\iint_{Q_n} (|\nabla u|^p + F^2 + 1) dx dt \right)^{\varkappa_n} + \gamma_n \iint_{Q_n} (F^2 + 1) dx dt.$$

This proves (2.6) for $\alpha = \varkappa_n - 1$, $n \in \mathbb{N}$. For a general $\alpha > 0$ fix n such that $\varkappa_n > \alpha + 1 > \varkappa_{n-1}$. Then there exists $s \in (0, 1)$ such that $p + 2\alpha = s(p + 2\varkappa_n - 2) + (1 - s)p$. Then

$$\iint_{Q_1} |\nabla u|^{p+2\alpha} dx dt \leq \left(\iint_{Q_1} |\nabla u|^{p+2\varkappa_n-2} dx dt \right)^s \left(\iint_{Q_1} |\nabla u|^p dx dt \right)^{1-s}.$$

Now (2.6) follows from (2.8) and the Young inequality. \square

The following a priori estimate, mainly extracted from [18, Theorem 1.1], is a ground for the assumption in Proposition 2.3 that u_ε is locally bounded uniformly in ε .

Proposition 2.8. *Let $u_\varepsilon \in V(Q)$ be a solution to (2.1) in Q . Then, for every $Q' = B' \times (t'_1, t'_2) \Subset Q'' = B'' \times (t''_1, t''_2) \Subset Q$ and $\rho < \frac{1}{4} \min[1, \text{dist}(B', \partial B), \sqrt{t'_1 - t''_1}, \sqrt{t'_2 - t''_2}]$, there exists $\gamma_\rho > 0$ independent of ε such that*

$$\sup_{Q'} |u_\varepsilon| \leq \gamma_\rho \left(\iint_{Q''} |u_\varepsilon|^{p+\frac{1}{Np'}} dx dt \right)^{\frac{pN}{2pN+p-1}} + \gamma_\rho \sup_{t \in (t'_1, t'_2)} \left(\int_{B''} |u_\varepsilon|^2 dx \right)^{\frac{1}{2}} + \gamma_\rho \sup_{x \in B''} W^{g^p+f}(x, 2\rho) + \gamma_\rho.$$

Proof. The fact that $u \in L^\infty_{loc}(Q)$ is established in [18, Theorem 1.1]. The actual estimate follows from [18, (3.18)]. \square

The next proposition establishes the existence of a solution to a initial-boundary value problem for (2.1).

Proposition 2.9. *Let $v \in V(\Omega_T) \cap L^{p'}_{loc}((0, T); W^{-1,p'}_{loc}(\Omega))$. Then there exists a solution $u_\varepsilon \in L^p((t_1, t_2); W^{1,p}(B_R))$ to (2.1) on Q subject to the condition $u_\varepsilon|_{\mathcal{P}Q} = v$, where $\mathcal{P}Q$ is the parabolic boundary of Q .*

Proof. The assertion follows from [20]. \square

The following proposition establishes first a posteriori estimates for a solution to an initial-boundary value problem for (2.1).

Proposition 2.10. *Let u_ε be a weak solution to (2.1) in $Q \Subset \Omega_T$. Assume that there exists $v \in V(\Omega_T) \cap L^{p'}_{loc}((0, T); W^{-1,p'}_{loc}(\Omega))$ such that $u_\varepsilon(x, t) = v(x, t)$ on $\mathcal{P}Q$. Then, provided $\sup_{x \in B_R} W^{g^p+f}_p(x, 2R)$ is small enough, the following estimates hold: there exists γ independent of ε such that*

$$\begin{aligned} \sup_{\tau \in (t_1, t_2)} \int_{B_R} u^2(\tau) dx + \iint_Q |\nabla u_\varepsilon|^p dx dt &\leq \gamma \sup_{\tau \in (t_1, t_2)} \int_{B_R} v^2(\tau) dx + \gamma \iint_Q (|\nabla v|^p + f|v|^p + f) dx dt \\ &\quad + \gamma \|\partial_t v\|_{L^{p'}((t_1, t_2); W^{-1,p'}(B_R))}, \\ \iint_Q |u|^{p+\frac{2p}{N}} dx dt &\leq \gamma \iint_Q |v|^{p+\frac{2p}{N}} dx + \gamma \left(\sup_{\tau \in (t_1, t_2)} \int_{B_R} v^2(\tau) dx + \iint_Q (|\nabla v|^p + f|v|^p + f) dx dt \right. \\ &\quad \left. + \|\partial_t v\|_{L^{p'}((t_1, t_2); W^{-1,p'}(B_R))} \right)^{1+\frac{p}{N}}. \end{aligned}$$

Proof. Fix $(s, \tau) \Subset (t_1, t_2)$. Test (2.1) by $\xi = T_h(u_h - v_h)$ with $h < \min\{t_2 - \tau, s - t_1\}$, to obtain that

$$\begin{aligned} \frac{1}{2} \int_{B_R} [(u_h(\tau) - v_h(\tau))^2 - (u_h(s) - v_h(s))^2] dx + \int_s^\tau \int_{B_R} \langle T_h[\mathbf{A}(u, \nabla u)], \nabla(u_h - v_h) \rangle dx dt \\ = \int_s^\tau \int_{B_R} T_h[b(u, \nabla u)](u_h - v_h) dx dt + \int_s^\tau \int_{B_R} (u_h - v_h)(\partial_t v_h) dx dt. \end{aligned}$$

Note that

$$\left| \int_s^\tau \int_{B_R} (u_h - v_h)(\partial_t v_h) dx dt \right| \leq \left(\int_s^\tau \int_{B_R} |\nabla(u_h - v_h)|^p dx dt \right)^{\frac{1}{p}} \|\partial_t v_h\|_{L^{p'}((s, \tau); W^{-1, p'}(B_R))}.$$

So we can pass to the limit as $h \rightarrow 0$ and then to the limit as $s \rightarrow t_1$ to obtain that

$$\begin{aligned} & \frac{1}{2} \sup_{\tau \in (t_1, t_2)} \int_{B_R} (u(\tau) - v(\tau))^2 dx + \iint_Q \langle \mathbf{A}(u, \nabla u), \nabla u \rangle dx dt \\ & \leq \iint_Q [|\mathbf{A}(u, \nabla u)| |\nabla v| + |b(u, \nabla u)| |u - v|] dx dt + \left(\iint_Q |\nabla(u - v)|^p dx dt \right)^{\frac{1}{p}} \|\partial_t v\|_{L^{p'}((t_1, t_2); W^{-1, p'}(B_R))}. \end{aligned}$$

Then, using structure conditions (1.2) and the Young inequality we obtain that, for all $\delta > 0$ there exists $\gamma > 0$ such that

$$\begin{aligned} & \sup_{\tau \in (t_1, t_2)} \int_{B_R} (u(\tau) - v(\tau))^2 dx + \iint_Q |\nabla u|^p dx dt \leq \delta \iint_Q |\nabla(u - v)|^p dx dt + \gamma \iint_Q g^p |u - v|^p dx dt \\ & + \gamma \iint_Q f |u - v|^p dx dt + \gamma \iint_Q f |v|^p dx dt \\ & + \gamma \iint_Q f dx dt + \delta^{-\frac{1}{p-1}} \gamma \|\partial_t v\|_{L^{p'}((t_1, t_2); W^{-1, p'}(B_R))}. \end{aligned}$$

The second and third terms on the right hand side are estimated by Corollary 1.10. Hence we have that

$$\begin{aligned} & \sup_{\tau \in (t_1, t_2)} \int_{B_R} (u(\tau) - v(\tau))^2 dx + \iint_Q |\nabla u - \nabla v|^p dx dt \leq \gamma \iint_Q |\nabla v|^p dx dt + \gamma \iint_Q f |v|^p dx dt \\ & + \gamma \iint_Q f dx dt + \gamma \|\partial_t v\|_{L^{p'}((t_1, t_2); W^{-1, p'}(B_R))}. \end{aligned}$$

Finally, by the Hölder and Sobolev inequalities we conclude that

$$\begin{aligned} & \iint_Q |u - v|^{p+\frac{2p}{N}} dx dt \leq \int_{t_1}^{t_2} \left(\int_{B_R} |u - v|^{\frac{pN}{N-p}} dx \right)^{\frac{N-p}{N}} \left(\int_{B_R} |u - v|^2 dx \right)^{\frac{p}{N}} dt \\ & \leq \gamma \iint_Q |\nabla u - \nabla v|^p dx dt \left(\sup_{\tau \in (t_1, t_2)} \int_{B_R} (u(\tau) - v(\tau))^2 dx \right)^{\frac{p}{N}}. \end{aligned}$$

□

The preceding proposition together with Proposition 2.8 turns the a priori estimate of Proposition 2.3 into an a posteriori one, as the following corollary states.

Corollary 2.11. *Let conditions of Proposition 2.10 be fulfilled. Assume that $(f^2 + f_1^2 + g_1^2 + f_2^2 + g_2^2)\mathbf{1}_{B_R} \in PK$. Then, for every $\alpha \geq 0$ and $l > 0$ there exist $\beta > 0$ and $\gamma_{l, \alpha} > 0$ independent of ε such that*

$$\operatorname{ess\,sup}_{t \in [t'_1, t'_2]} \int_{B'} |\nabla u_\varepsilon|^{2+2\alpha} dx + \iint_{Q'} \left| D(\nabla u_\varepsilon (|\nabla u_\varepsilon| - l)_+^{\alpha + \frac{p}{2} - 1}) \right|^2 dx d\tau \leq \gamma_{l, \alpha},$$

provided $(g_1^2 + g_2^2)\mathbf{1}_{B_R} \in PK_\beta$.

In the following proposition we prove that the solutions u_ε and certain functions of their gradients are locally Lipschitz continuous in time variable, with values in certain Banach spaces, uniformly in ε . This will be used to apply a compactness result from [25].

Proposition 2.12. *Let assumptions of Corollary 2.11 be fulfilled. Then for all $\sigma > N$, $\alpha \geq p$ and $l > 0$, $B' \Subset B_R$ and $(t'_1, t'_2) \Subset (t_1, t_2)$ there are constants $\beta, \gamma > 0$ independent of ε such that, if $(g_1^2 + g_2^2)\mathbf{1}_{B_R} \in PK_\beta$, then, for all $h \in (0, t'_2 - t'_1)$,*

$$\begin{aligned} \int_{t'_1}^{t'_2-h} \|u_\varepsilon(t+h) - u_\varepsilon(t)\|_{W^{-1, \frac{p}{p-1}}(B')} dt &\leq \gamma h; \\ \int_{t'_1}^{t'_2-h} \|(|\nabla u_\varepsilon(t+h)| - l)_+^\alpha \nabla u_\varepsilon(t+h) - (|\nabla u_\varepsilon(t)| - l)_+^\alpha \nabla u_\varepsilon(t)\|_{W^{-1, \frac{\sigma}{\sigma-1}}(B')} dt &\leq \gamma h. \end{aligned}$$

Proof. The first assertion follows from Propositions 2.1 and 2.10.

To shorten the proof of the second assertion, we introduce some notation. Let $F = g + g_1 + g_2 + f + f_1 + f_2$. For $l > 0$ and $\alpha \geq 2$ define $w_{l,\alpha} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ as follows. For $\zeta \in \mathbb{R}^N$ let $w(\zeta) \equiv w_{l,\alpha}(\zeta) = (|\zeta| - l)_+^\alpha \zeta$. Also, we define $\mathfrak{h}_u : Q \rightarrow (\mathbb{R}^{N \times N})^*$ as follows. For $M \in \mathbb{R}^{N \times N}$,

$$\mathfrak{h}_u[M] := \text{tr} \{ M(\partial_z \mathbf{A} D^2 u + \partial_x \mathbf{A}) \} + \langle M \partial_u \mathbf{A}, \nabla u \rangle + b(u, \nabla u) \text{tr} M.$$

Then it follows from Lemma 2.4 that, for any $\tau \in (t_1, t_2 - h)$

$$\int \langle \nabla u, \zeta \rangle dx \Big|_\tau^{\tau+h} - \int_\tau^{\tau+h} \int_B \langle \nabla u, \partial_t \zeta \rangle dx dt = \int_\tau^{\tau+h} \int_B \mathfrak{h}_u[D\zeta] dx dt \quad \text{for all } \zeta \in H_c^1(Q \rightarrow \mathbb{R}^N).$$

Recall that we have to verify that, for all $B' \Subset B_R$, $(t'_1, t'_2) \Subset (t_1, t_2)$ and $h \in (0, t'_2 - t'_1)$,

$$\begin{aligned} &\int_{t'_1}^{t'_2-h} \|w(\nabla u(\tau+h)) - w(\nabla u(\tau))\|_{W^{-1, \sigma'}(B')} dt \\ &\equiv \int_{t'_1}^{t'_2-h} \sup_{\|\zeta\|_{W_0^{1, \sigma}(B')} \leq 1} \left| \int \langle w(\nabla u(\tau+h)) - w(\nabla u(\tau)), \zeta \rangle dx \right| dt \leq \gamma h, \end{aligned}$$

with some $\gamma > 0$ independent of $h > 0$ and u .

Note that, for a vector field ζ differentiable in t , one has that $w(\zeta)$ is differentiable in t and

$$\partial_t w(\zeta) = (|\zeta| - l)_+^\alpha \partial_t \zeta + \alpha \frac{\zeta}{|\zeta|} (|\zeta| - l)_+^{\alpha-1} \langle \zeta, \partial_t \zeta \rangle.$$

Hence by using the Steklov averaging one obtains that, for all $\zeta \in C_c^1(B' \rightarrow \mathbb{R}^N)$,

$$(2.9) \quad \int \langle w(\nabla u), \zeta \rangle dx \Big|_\tau^{\tau+h} = \int_\tau^{\tau+h} \int_{B_R} \mathfrak{h}_u[D\tilde{\zeta}_1 + D\tilde{\zeta}_2] dx dt,$$

with $\tilde{\zeta}_1 = (|\nabla u| - l)_+^\alpha \zeta$ and $\tilde{\zeta}_2 = \alpha \frac{\nabla u}{|\nabla u|} (|\nabla u| - l)_+^{\alpha-1} \langle \nabla u, \zeta \rangle$. So, for $B' \Subset B_R$, $(t'_1, t'_2) \Subset (t_1, t_2)$ and $h \in (0, t_2 - t'_2)$, $\zeta \in C_c^1(B' \rightarrow \mathbb{R}^N)$, we have

$$\int_{t'_1}^{t'_2-h} \sup_{\|\zeta\|_{W_0^{1, \sigma}(B')} \leq 1} \left| \int \langle w(\nabla u(t+h)) - w(\nabla u(t)), \zeta \rangle dx \right| dt \leq h \int_{t'_1}^{t'_2} \sup_{\|\zeta\|_{W_0^{1, \sigma}(B')} \leq 1} \int_{B'} |\mathfrak{h}_u[D\tilde{\zeta}_1 + D\tilde{\zeta}_2]| dx dt.$$

Now observe that, by assumptions (1.2) and (1.6)–(1.9), for every $M \in \mathbb{R}^{N \times N}$,

$$(2.10) \quad \begin{aligned} |\mathfrak{h}_u[M]| &\leq \gamma |M| \{ (|\nabla u|^{p-2} + 1) |D^2 u| + (g + g_2) |\nabla u|^{p-1} + g_1 |\nabla u|^{p-2} + f + f_1 + f_2 \} \\ &\leq \gamma_{p,l} |M| \{ (|\nabla u| - l)_+^{p-2} + 1 \} |D^2 u| + F[(|\nabla u| - l)_+^{p-1} + 1]. \end{aligned}$$

In turn, we compute that

$$D\tilde{\zeta}_1 = (|\nabla u| - l)_+^\alpha D\zeta + \alpha \frac{\nabla u}{|\nabla u|} (|\nabla u| - l)_+^{\alpha-1} \zeta \otimes \left(D^2 u \frac{\nabla u}{|\nabla u|} \right)$$

and

$$\begin{aligned} D\tilde{\zeta}_2 = & \alpha \frac{\nabla u}{|\nabla u|} (|\nabla u| - l)_+^{\alpha-1} \frac{\nabla u}{|\nabla u|} \otimes \{ (D\zeta)^\top \nabla u + D^2 u \zeta \} \\ & + \alpha \frac{\nabla u}{|\nabla u|} (|\nabla u| - l)_+^{\alpha-1} \langle \nabla u, \zeta \rangle \left(\frac{D^2 u}{|\nabla u|} - \frac{\nabla u}{|\nabla u|} \otimes \left(\frac{D^2 u}{|\nabla u|} \frac{\nabla u}{|\nabla u|} \right) \right) \\ & + \alpha(\alpha - 1) \frac{\nabla u}{|\nabla u|} (|\nabla u| - l)_+^{\alpha-2} \langle \nabla u, \zeta \rangle \frac{\nabla u}{|\nabla u|} \otimes \left(D^2 u \frac{\nabla u}{|\nabla u|} \right). \end{aligned}$$

Hence

$$\begin{aligned} (2.11) \quad |D\tilde{\zeta}_1 + D\tilde{\zeta}_2| & \leq \gamma_\alpha (|D\zeta| |\nabla u| (|\nabla u| - l)_+^{\alpha-1} + |\zeta| |D^2 u| |\nabla u| (|\nabla u| - l)_+^{\alpha-2}) \\ & \leq \gamma_{\alpha, l} \{ |D\zeta| [(|\nabla u| - l)_+^\alpha + (|\nabla u| - l)_+^{\alpha-1}] + |\zeta| |D^2 u| [(|\nabla u| - l)_+^{\alpha-1} + (|\nabla u| - l)_+^{\alpha-2}] \}. \end{aligned}$$

So it follows from (2.9)–(2.11) that

$$\begin{aligned} \frac{1}{\gamma_{\alpha, p, l}} \int_{B'} |\mathfrak{h}_u [D\tilde{\zeta}_1 + D\tilde{\zeta}_2]| dx & \leq \int_{B'} |\zeta| |D^2 u|^2 \left[(|\nabla u| - l)_+^{p+\alpha-3} + (|\nabla u| - l)_+^{\alpha-2} \right] dx \\ & \quad + \int_{B'} |\zeta| |D^2 u| F \left[(|\nabla u| - l)_+^{p+\alpha-2} + (|\nabla u| - l)_+^{\alpha-1} \right] dx \\ & \quad + \int_{B'} |D\zeta| |D^2 u| \left[(|\nabla u| - l)_+^{p+\alpha-2} + (|\nabla u| - l)_+^{\alpha-1} \right] dx \\ & \quad + \int_{B'} |D\zeta| F \left[(|\nabla u| - l)_+^{p+\alpha-1} + (|\nabla u| - l)_+^{\alpha-1} \right] dx. \end{aligned}$$

Now it follows from the Hölder inequality that

$$\begin{aligned} \frac{1}{\gamma_{\alpha, p, l}} \int_{B'} |\mathfrak{h}_u [D\tilde{\zeta}_1 + D\tilde{\zeta}_2]| dx & \leq \|\zeta\|_\infty \int_{B'} |D^2 u|^2 \left[(|\nabla u| - l)_+^{p+\alpha-3} + (|\nabla u| - l)_+^{\alpha-2} \right] dx \\ & \quad + \|\zeta\|_\infty \left(\int_{B'} F^2 dx \right)^{\frac{1}{2}} \left(\int_{B'} |D^2 u|^2 \left[(|\nabla u| - l)_+^{2p+2\alpha-4} + (|\nabla u| - l)_+^{2\alpha-4} \right] dx \right)^{\frac{1}{2}} \\ & \quad + \|D\zeta\|_2 \left(\int_{B'} |D^2 u|^2 \left[(|\nabla u| - l)_+^{2p+2\alpha-4} + (|\nabla u| - l)_+^{2\alpha-2} \right] dx \right)^{\frac{1}{2}} \\ & \quad + \|D\zeta\|_N \left(\int_{B'} F^2 dx \right)^{\frac{1}{2}} \left[\int_{B'} (|\nabla u| - l)_+^{(p+\alpha-1)\frac{2N}{N-2}} + \int_{B'} (|\nabla u| - l)_+^{(\alpha-1)\frac{2N}{N-2}} \right]^{\frac{N-2}{2N}}. \end{aligned}$$

Thus,

$$\int_{B'} |\mathfrak{h}_u [D\tilde{\zeta}_1 + D\tilde{\zeta}_2]| dx \leq c(u) (\|\zeta\|_\infty + \|D\zeta\|_2 + \|D\zeta\|_N)$$

with

$$\begin{aligned} c(u) = & \gamma_{\alpha, p, l} \left\{ \int_{B'} |D^2 u|^2 \left[(|\nabla u| - l)_+^{2p+2\alpha-4} + (|\nabla u| - l)_+^{\alpha-2} \right] dx \right. \\ & \left. + \left(\int_{B'} [|\nabla u|^{p+\alpha-1} + |\nabla u|^{\alpha-1}]^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} + \int_{B'} F^2 dx \right\}. \end{aligned}$$

Finally, by the Sobolev embedding theorem, for any $\sigma > N$, one has $\|\zeta\|_\infty + \|D\zeta\|_2 + \|D\zeta\|_N \leq c\|\zeta\|_{W_0^{1,\sigma}(B')}$, and by Corollary 2.11 $c(u)$ is bounded by a constant independent of u provided $\alpha \geq p$. So the second assertion follows. \square

The following lemma serves to assert the pointwise convergence of the gradient.

Lemma 2.13. *Let ξ_n be a sequence of a.e. finite vector fields such that there exists $\alpha > 0$ such that $\xi_n(|\xi_n| - \frac{1}{m})_+^\alpha$ converges a.e. as $n \rightarrow \infty$ for all $m \in \mathbb{N}$. Then ξ_n converges a.e. as $n \rightarrow \infty$.*

Proof. Denote $\eta_n := \xi_n(|\xi_n| - \frac{1}{m})_+^\alpha$ and $E_{nm} := \{x : |\xi_n| \geq \frac{1}{m}\}$.

Note that the function $\phi_m(s) := s(s - \frac{1}{m})_+^\alpha$ is a homeomorphism $[\frac{1}{m}, \infty) \rightarrow [0, \infty)$. Let ψ_m denote the inverse map. Then one has

$$\xi_n \chi_{E_{nm}} = \eta_n \frac{\psi_m(|\eta_n|)}{|\eta_n|}.$$

So there are vector fields ζ_m , $m \in \mathbb{N}$ such that $\xi_n \chi_{E_{nm}} \rightarrow \zeta_m$ a.e. as $n \rightarrow \infty$.

Let

$$E_m = \liminf_{n \rightarrow \infty} E_{nm} = \bigcup_{N \in \mathbb{N}} \bigcap_{n \geq N} E_{nm} = \{x : \liminf_{n \rightarrow \infty} |\xi_n|(x) \geq \frac{1}{m}\}, \quad E := \bigcup_{m \in \mathbb{N}} E_m = \{x : \liminf_{n \rightarrow \infty} |\xi_n|(x) > 0\}.$$

Then, for every $x \in E_m$ there exists $N \in \mathbb{N}$ such that $x \in E_{nm}$ for all $n \geq N$. Hence

$$\lim_{n \rightarrow \infty} \xi_n(x) = \lim_{n \rightarrow \infty} \xi_n(x) \chi_{E_{nm}} = \zeta_m(x) \text{ for all } x \in E_m.$$

Thus $\xi_n(x) \rightarrow \xi(x)$ as $n \rightarrow \infty$ for all $x \in E$. Note that $|\xi|(x) \geq \frac{1}{m}$ for all $x \in E_m$. So $\zeta_m(x) = \xi \chi_{\{|\xi| \geq \frac{1}{m}\}}(x)$ for all $m \in \mathbb{N}$ and $x \in E$.

Further,

$$E^c := \{x : \forall m, N \in \mathbb{N} \exists n \geq N \text{ such that } |\xi_n|(x) < \frac{1}{m}\} = \{x : \liminf_{n \rightarrow \infty} |\xi_n|(x) = 0\}.$$

Therefore, for all $x \in E^c$ and $m \in \mathbb{N}$,

$$|\zeta_m|(x) = \liminf_{n \rightarrow \infty} |\xi_n| \chi_{E_{nm}}(x) = 0.$$

Now we define $\xi(x) = 0$ for $x \in E^c$ so that $\zeta_m(x) = \xi \chi_{\{|\xi| \geq \frac{1}{m}\}}(x)$ a.e. Finally, we have

$$\limsup_{n \rightarrow \infty} |\xi_n - \xi| \leq \limsup_{n \rightarrow \infty} |\xi_n \chi_{E_{nm}} - \zeta_m| + \limsup_{n \rightarrow \infty} |\xi_n \chi_{\{|\xi_n| < \frac{1}{m}\}} + |\xi \chi_{\{|\xi| < \frac{1}{m}\}}| < \frac{2}{m} \rightarrow 0 \text{ as } m \rightarrow \infty.$$

\square

Proof of Theorem 1.1. For $\varepsilon > 0$, let \mathbf{A}_ε , b_ε , and u_ε be as in Proposition 2.9. Let $Q' := (t'_1, t'_2) \times B' \Subset Q$. Due to the embedding $W^{1,q}(B') \Subset L^1(B') \subset W^{-1, \frac{\sigma}{\sigma-1}}(B')$ for any $q \geq 1$ and $\sigma > N$, it follows from [25, Theorem 5] and Corollary 2.11 and Proposition 2.12 that, for any $\alpha \geq p$, $l > 0$, the sets $\{u_\varepsilon\}_{\varepsilon > 0}$ and $\{(|\nabla u_\varepsilon| - l)_+^\alpha \nabla u_\varepsilon\}_{\varepsilon > 0}$ are compact in $L^1(Q')$. Using a compact exhaustion of Q and a standard diagonalization we conclude that there exists a subsequence $\varepsilon_n \downarrow 0$ such that, $u_n = u_{\varepsilon_n}$ converges as $n \rightarrow \infty$ a.e. on Q and $\nabla u_n(|\nabla u_n| - \frac{1}{m})_+^\alpha$ converges as $n \rightarrow \infty$ a.e. on Q for all $m \in \mathbb{N}$. Then by Lemma 2.13 it follows that ∇u_n converges as $n \rightarrow \infty$ a.e. on Q . Let u denote the pointwise limits of u_n . Since ∇u_n is uniformly bounded in $L_{loc}^q(Q)$ for all $q > 1$, we conclude that $\nabla u \in L_{loc}^q(Q)$ for all $q > 1$ and $\nabla u_n \rightarrow \nabla u$ as $n \rightarrow \infty$ weakly in $L_{loc}^q(Q)$. Since the weak and the pointwise limits coincide, $\nabla u_n \rightarrow \nabla u$ as $n \rightarrow \infty$ a.e. on Q .

Now observe that

$$\begin{aligned} |\mathbf{A}_{\varepsilon_n}(u_n, \nabla u_n) - \mathbf{A}(u, \nabla u)| &\leq |\mathbf{A}(u_n, \nabla u_n) - \mathbf{A}(u, \nabla u)| + |u_n - u| \int_0^1 |\partial_u \mathbf{A}(\omega_s)| ds \\ &+ |\nabla u_n - \nabla u| \int_0^1 |\partial_z \mathbf{A}(\omega_s)| ds, \quad \text{where } \omega_s = ((1-s)u_n + su, (1-s)\nabla u_n + s\nabla u). \end{aligned}$$

Using the structure conditions (1.7), (1.8) we infer that $\mathbf{A}_{\varepsilon_n}(u_n, \nabla u_n) \rightarrow \mathbf{A}(u, \nabla u)$ as $n \rightarrow \infty$ a.e. on Q and that, due to (1.2), the set $\{\mathbf{A}_{\varepsilon}(u_{\varepsilon}, \nabla u_{\varepsilon})\}_{\varepsilon>0}$ is bounded in $L_{loc}^{\frac{p}{p-1}}(Q)$. Hence $\mathbf{A}_{\varepsilon_n}(u_n, \nabla u_n) \rightarrow \mathbf{A}(u, \nabla u)$ as $n \rightarrow \infty$ weakly in $L_{loc}^{\frac{p}{p-1}}(Q)$. Now we note that

$$\begin{aligned} |b_{\varepsilon_n}(u_n, \nabla u_n) - b(u, \nabla u)| &\leq |b(u_n, \nabla u_n) - b(u, \nabla u)| \\ &+ (\mathbf{1}_{\{|b(u_n, \nabla u_n) - b(u, \nabla u)| \geq 1/2\}} + \mathbf{1}_{\{|b(u, \nabla u)| > 1/(2\varepsilon_n)\}})(|b(u_n, \nabla u_n)| + |b(u, \nabla u)|). \end{aligned}$$

Hence, due to (1.2) $b_{\varepsilon_n}(u_n, \nabla u_n) \rightarrow b(u, \nabla u)$ a.e. on Q . Then by (1.2), the set $\{b_{\varepsilon_n}(u_n, \nabla u_n)\}$ is bounded in $L_{loc}^{\frac{p}{p-1}}(Q)$. So $b_{\varepsilon_n}(u_n, \nabla u_n)$ is weakly compact in $L_{loc}^{p'}(Q)$ and hence $b(u, \nabla u) \in L_{loc}^{p'}(Q)$ and $b_{\varepsilon_n}(u_n, \nabla u_n) \rightarrow b(u, \nabla u)$ as $n \rightarrow \infty$ weakly in $L_{loc}^{p'}(Q)$. Hence, for every $\theta \in W_c^{1,p}(B)$, we have that $\iint_Q \mathbf{A}_{\varepsilon_n}(u_n, \nabla u_n) \nabla \theta dx d\tau \rightarrow \iint_Q \mathbf{A}(u, \nabla u) \nabla \theta dx d\tau$ and $\iint_Q b_{\varepsilon_n}(u_n, \nabla u_n) \theta dx d\tau \rightarrow \iint_Q b(u, \nabla u) \theta dx d\tau$ as $n \rightarrow \infty$. Thus u is a solution to (1.1) satisfying estimate (1.10). \square

3 Proof of Theorem 1.2

In the proof we follow the idea from [27], with required modifications. We start with the following technical lemma.

Lemma 3.1. *There exist $c_p, \Gamma_p > 0$ such that, for all $(x, t) \in \Omega_T$, $\mu, \tilde{\mu} \in \mathbb{R}$, $\eta, \tilde{\eta} \in \mathbb{R}^N$, one has*

$$\begin{aligned} (3.1) \quad &\langle \mathbf{A}(x, t, \mu, \eta) - \mathbf{A}(x, t, \tilde{\mu}, \tilde{\eta}), \eta - \tilde{\eta} \rangle \geq c_p(|\eta| + |\tilde{\eta}|)^{p-2} |\eta - \tilde{\eta}|^2 \\ &- \Gamma_p \left(f_1^{p'}(x) |\mu - \tilde{\mu}|^{p'} + g_1^2(x) |\tilde{\eta}|^{p-2} |\mu - \tilde{\mu}|^2 + g_1^2(x) |\eta - \tilde{\eta}|^{p-2} |\mu - \tilde{\mu}|^2 \right). \end{aligned}$$

Proof. Set $\omega_s := (x, t, s\mu + (1-s)\tilde{\mu}, s\eta + (1-s)\tilde{\eta})$, $s \in [0, 1]$. Then

$$\mathbf{A}(x, t, \mu, \eta) - \mathbf{A}(x, t, \tilde{\mu}, \tilde{\eta}) = \int_0^1 \partial_z \mathbf{A}(\omega_s) (\eta - \tilde{\eta}) ds + \int_0^1 \partial_u \mathbf{A}(\omega_s) (\mu - \tilde{\mu}) ds.$$

Then, by (1.6), there exist $c_{0,p} > 0$ such that

$$\int_0^1 \langle \partial_z \mathbf{A}(\omega_s) (\eta - \tilde{\eta}), \eta - \tilde{\eta} \rangle ds \geq c_{0,p} (|\eta| + |\tilde{\eta}|)^{p-2} |\eta - \tilde{\eta}|^2.$$

Further, by (1.8), there exists C_p such that

$$\begin{aligned} \left| \int_0^1 \langle \partial_u \mathbf{A}(\omega_s) (\mu - \tilde{\mu}), \eta - \tilde{\eta} \rangle ds \right| &\leq f_1(x) |\mu - \tilde{\mu}| |\eta - \tilde{\eta}| + C_p g_1(x) (|\eta| + |\tilde{\eta}|)^{p-2} |\mu - \tilde{\mu}| |\eta - \tilde{\eta}| \\ &\leq \frac{c_{0,p}}{4} |\eta - \tilde{\eta}|^p + \frac{1}{c_{0,p}} f_1^{p'}(x) |\mu - \tilde{\mu}|^{p'} + \frac{c_{0,p}}{4} (|\eta| + |\tilde{\eta}|)^{p-2} |\eta - \tilde{\eta}|^2 \\ &\quad + \frac{C_p^2}{c_{0,p}} g_1^2(x) (|\eta| + |\tilde{\eta}|)^{p-2} |\mu - \tilde{\mu}|^2 \\ &\leq \frac{c_{0,p}}{2} (|\eta| + |\tilde{\eta}|)^{p-2} |\eta - \tilde{\eta}|^2 + \frac{1}{c_{0,p}} f_1^{p'}(x) |\mu - \tilde{\mu}|^{p'} \\ &\quad + \frac{2^{p-2} C_p^2}{c_{0,p}} g_1^2(x) |\eta - \tilde{\eta}|^{p-2} |\mu - \tilde{\mu}|^2 + \frac{4^{p-2} C_p^2}{c_{0,p}} g_1^2(x) |\tilde{\eta}|^{p-2} |\mu - \tilde{\mu}|^2. \quad \square \end{aligned}$$

Similar to what was done in [27] we introduce the following functions:

$$\begin{aligned} \widehat{b}(x, t, \tilde{\mu}, \tilde{\eta}) &:= \Gamma_p \left(f_1^{\frac{p}{p-1}}(x) |u(x, t) - \tilde{\mu}|^{\frac{2-p}{p-1}} + g_1^2(x) |\tilde{\eta}|^{p-2} \right) (u(x, t) - \tilde{\mu}), \\ \overline{b}(x, t, \tilde{\mu}, \tilde{\eta}) &:= (-f(x)(1 + |u(x, t)|^{p-1}) - g(x)|2\tilde{\eta}|^{p-1}) \vee b(x, t, u(x, t), \nabla u(x, t)) \wedge \\ &\quad \wedge (f(x)(1 + |u(x, t)|^{p-1}) + g(x)|2\tilde{\eta}|^{p-1}). \end{aligned}$$

Set

$$\tilde{b}(x, t, \tilde{\mu}, \tilde{\eta}) = \hat{b}(x, \tilde{\mu}, \tilde{\eta}) + \bar{b}(x, \tilde{\mu}, \tilde{\eta}).$$

Consider the auxiliary the equation

$$(3.2) \quad \partial_t \tilde{u} - \operatorname{div} \mathbf{A}(\tilde{u}, \nabla \tilde{u}) = \tilde{b}(\tilde{u}, \nabla \tilde{u}).$$

Proposition 3.2. *Let $Q = B_R \times (t_1, t_2) \Subset \Omega_T$. Let \tilde{u} be a weak solution to (3.2) in Q such that*

$$u|_{\mathcal{P}Q} = \tilde{u}|_{\mathcal{P}Q},$$

where $\mathcal{P}Q$ is the parabolic boundary of Q . Then $\tilde{u} = u$ in Q if $\sup_{x \in B_R} W_p^{g^p}(x, 2R)$ and $\sup_{x \in B_R} W_p^{g_1^p}(x, 2R)$ are small enough.

Proof. Subtract (3.2) out of (1.1) and multiply the difference by $u - \tilde{u}$. Note that the latter belongs to $L^p((t_1, t_2) \rightarrow W_0^{1,p}(B)) \cap C_0([t_1, t_2] \rightarrow L^2(B))$. We obtain that

$$\frac{1}{2} \int_B |u - \tilde{u}|^2(t_2) dx + \iint_Q \langle \mathbf{A}(u, \nabla u) - \mathbf{A}(\tilde{u}, \nabla \tilde{u}), \nabla u - \nabla \tilde{u} \rangle dx dt = \iint_Q (b(u, \nabla u) - \tilde{b}(\tilde{u}, \nabla \tilde{u}))(u - \tilde{u}) dx dt.$$

By Lemma 3.1 we have

$$\begin{aligned} & \iint_Q \langle \mathbf{A}(u, \nabla u) - \mathbf{A}(\tilde{u}, \nabla \tilde{u}), \nabla u - \nabla \tilde{u} \rangle dx dt + \iint_Q \hat{b}(\tilde{u}, \nabla \tilde{u})(u - \tilde{u}) dx dt \\ & \geq c_p \iint_Q (|\nabla u| + |\nabla \tilde{u}|)^{p-2} |\nabla u - \nabla \tilde{u}|^2 dx dt - \Gamma_p \|\nabla u - \nabla \tilde{u}\|_p^{p-2} \|g_1(u - \tilde{u})\|_p^2. \end{aligned}$$

Further, note that $b(u, \nabla u)$ is of the same sign that $\bar{b}(\tilde{u}, \nabla \tilde{u})$. Also observe that $b(u, \nabla u) \neq \bar{b}(\tilde{u}, \nabla \tilde{u})$ only under the condition $|b(u, \nabla u)| > f(1 + |u|^{p-1}) + g|2\nabla \tilde{u}|^{p-1}$, which implies that $|\nabla u| \geq 2|\nabla \tilde{u}|$. Hence

$$|b(u, \nabla u) - \bar{b}(\tilde{u}, \nabla \tilde{u})| \leq g|\nabla u|^{p-1} \mathbf{1}_{\{|\nabla u| \geq 2|\nabla \tilde{u}|\}} \leq 2^{p-1} g |\nabla u - \nabla \tilde{u}|^{p-1}.$$

Therefore

$$\iint_Q |b(u, \nabla u) - \bar{b}(\tilde{u}, \nabla \tilde{u})| |u - \tilde{u}| dx dt \leq 2^{p-1} \|\nabla u - \nabla \tilde{u}\|_p^{p-1} \|g(u - \tilde{u})\|_p^p.$$

Thus we obtain that

$$c_p \iint_Q (|\nabla u| + |\nabla \tilde{u}|)^{p-2} |\nabla u - \nabla \tilde{u}|^2 dx dt \leq \Gamma_p \|\nabla u - \nabla \tilde{u}\|_p^{p-2} \|g_1(u - \tilde{u})\|_p^2 + 2^{p-1} \|\nabla u - \nabla \tilde{u}\|_p^{p-1} \|g(u - \tilde{u})\|_p^p.$$

By (1.22) this implies that

$$c_p \|\nabla u - \nabla \tilde{u}\|_p^p \leq \left\{ \Gamma_p \sup_{B_R} \left(W_p^{g_1^p}(x, 2R) \right)^{\frac{2}{p'}} + 2^{p-1} \sup_{B_R} \left(W_p^{g^p}(x, 2R) \right)^{\frac{1}{p'}} \right\} \|\nabla u - \nabla \tilde{u}\|_p^p.$$

So if $\sup_{B_R} W_p^{g^p}(x, 2R)$ and $\sup_{B_R} W_p^{g_1^p}(x, 2R)$ are small enough then $\|\nabla u - \nabla \tilde{u}\|_p^p \leq 0$. \square

Proof of Theorem 1.2. Note that $|\bar{b}(x, t, \tilde{\mu}, \tilde{\eta})| \leq f(x) + g(x)|2\tilde{\eta}|^{p-1}$ and

$$|\hat{b}(x, t, \tilde{\mu}, \tilde{\eta})| \leq \Gamma_p \left(g_1(x) |\tilde{\eta}|^{p-1} + (f_1(x)^{\frac{p}{p-1}} + g_1(x)^p) |u(x, t) - \tilde{\mu}|^{p-1} + f_1(x)^{\frac{p}{p-1}} \right).$$

Hence equation (3.2) satisfies the structural conditions (1.2)-(1.9) with $2^{p-1} \Gamma_p (f_1^{\frac{p}{p-1}} + g_1^p + \sup |u|) + f$ and $(2^{p-1} g + \Gamma_p g_1)$ replacing f and g , respectively. Therefore by Theorem 1.1, there exists a solution \tilde{u} coinciding with u on the parabolic boundary of Q , which enjoys the estimate (1.10). Since $g^p \in K_p$, we can choose R so small that $u = \tilde{u}$ on Q . Hence the assertion follows. \square

4 Boundedness of the gradient. Proof of Theorem 1.3

We obtain uniform estimates of the gradients on the sets where $|\nabla u| > l$ for some positive l . This restriction allows us to simplify the structure conditions putting $F = f + g + f_1 + g_1 + f_2 + g_2$ and requiring

$$(4.1) \quad |\partial_x \mathbf{A}| + |\partial_u \mathbf{A}||z| + |b| \leq F(x)|z|^{p-1}$$

instead of the last condition in (1.2) and (1.8), (1.9). In obtaining the estimates we follow the parabolic version of the Kilpeläinen–Malý technique [12, 21] (see [18, 26]).

Let $\lambda > 0$, $\delta > 0$, $l \geq l_0 \geq 1$, $\sigma := \left(\frac{|\nabla u|^2 - l}{\delta}\right)_+$. Set

$$\varphi(\sigma) := \int_0^\sigma (1+s)^{-1-\lambda} ds, \quad G(\sigma) = \int_0^\sigma s\varphi(s)ds.$$

Before formulating the next lemma let us note that $G(\sigma) \asymp \min\{\sigma^2, \sigma^3\}$, $\sigma > 0$ and

$$(4.2) \quad \varphi(\sigma) \asymp \frac{\sigma}{\sigma+1} \geq \frac{\sigma}{(1+\sigma)^{1+\lambda}} \quad \text{so that} \quad \partial_\sigma(\sigma\varphi(\sigma)) \asymp \varphi(\sigma).$$

Lemma 4.1. *Let $\delta > 0$. With notation $w = |\nabla u|^2$, $\sigma = \left(\frac{w-l}{\delta}\right)_+$ the following inequality holds*

$$(4.3) \quad \begin{aligned} \operatorname{ess\,sup}_t \int G(\sigma(t))\xi^q(t) + \iint w^{\frac{p}{2}-1} |\nabla \sigma|^2 \varphi(\sigma)\xi^q &\leq \gamma \iint \sigma^2 \xi^{q-1} |\partial_t \xi| + \gamma \iint w^{\frac{p}{2}-1} |\nabla \xi|^2 \sigma^2 \varphi(\sigma)\xi^{q-2} \\ &+ \gamma \delta^{-2} l^{\frac{p}{2}+1} \iint F^2 \xi^q + \gamma \delta^{\frac{p}{2}-1} \iint F^2 \sigma^{\frac{p}{2}+1} \varphi(\sigma)\xi^q. \end{aligned}$$

Proof. Due to the last assertion of Theorem 1.1, we may assume that $u \in L_{loc}^2((0, T); W_{loc}^{2,2}(\Omega)) \cap C((0, T); W_{loc}^{1,2}(\Omega))$. By Lemmas 2.4 and 2.6 with $\zeta = (\nabla u)\sigma\varphi(\sigma)\xi^q$ we have

$$(4.4) \quad \frac{\delta}{2} \int_\Omega G(\sigma(t))\xi^q(t) + \iint_{(0,t) \times \Omega} \operatorname{tr}\{(D\zeta)(\partial_z \mathbf{A})D^2 u\} \leq q \frac{\delta}{2} \iint_{(0,t) \times \Omega} G(\sigma)\xi^{q-1} \partial_\tau \xi + \iint_{(0,t) \times \Omega} (|\partial_x \mathbf{A}| + |\partial_u \mathbf{A}||\nabla u| + |b|)|D\zeta|.$$

Note that $D\zeta = D^2 u \sigma \varphi(\sigma)\xi^q + (\nabla u \otimes \nabla \sigma)(\varphi(\sigma) + \sigma(1+\sigma)^{-1-\lambda})\xi^q + q(\nabla u \otimes \nabla \xi)\sigma\varphi(\sigma)\xi^{q-1}$, and $D^2 u \nabla u = \frac{1}{2} \nabla w = \delta/2 \nabla \sigma$. By structure conditions (1.6) and (1.7), Remark 2.5 and (4.2) it follows that

$$\operatorname{tr}\{(D\zeta)(\partial_z \mathbf{A})D^2 u\} \geq c_0 w^{\frac{p}{2}-1} |D^2 u|_{HS}^2 \sigma \varphi(\sigma)\xi^q + \frac{c_0}{2} \delta w^{\frac{p}{2}-1} |\nabla \sigma|^2 \varphi(\sigma)\xi^q - \frac{c_1}{2} \delta w^{\frac{p}{2}-1} |\nabla \sigma| |\nabla \xi| \sigma \varphi(\sigma)\xi^{q-1}.$$

By the Schwartz inequality

$$\operatorname{tr}\{(D\zeta)(\partial_z \mathbf{A})D^2 u\} \geq c_0 w^{\frac{p}{2}-1} |D^2 u|_{HS}^2 \sigma \varphi(\sigma)\xi^q + \frac{c_0}{4} \delta w^{\frac{p}{2}-1} |\nabla \sigma|^2 \varphi(\sigma)\xi^q - \frac{c_1^2}{4c_0} \delta w^{\frac{p}{2}-1} |\nabla \xi|^2 \sigma^2 \varphi(\sigma)\xi^{q-2}.$$

To estimate the right hand side of (4.4) we note that

$$(4.5) \quad \begin{aligned} &(|\partial_x \mathbf{A}| + |\partial_u \mathbf{A}||\nabla u| + |b|)|D\zeta| \\ &\leq F w^{\frac{p-1}{2}} |D^2 u| \sigma \varphi(\sigma)\xi^q + 2F w^{\frac{p}{2}} |\nabla \sigma| \varphi(\sigma)\xi^q + qF w^{\frac{p}{2}} |\nabla \xi| \sigma \varphi(\sigma)\xi^{q-1} \\ &\leq \frac{c_0}{16} w^{\frac{p}{2}-1} |D^2 u|^2 \sigma \varphi(\sigma)\xi^q + \frac{c_0}{16} \delta w^{\frac{p}{2}-1} |\nabla \sigma|^2 \varphi(\sigma)\xi^q \\ &\quad + \gamma \delta w^{\frac{p}{2}-1} |\nabla \xi|^2 \sigma^2 \varphi(\sigma)\xi^{q-2} + \frac{\gamma}{\delta} F^2 w^{\frac{p}{2}+1} \varphi(\sigma)\xi^q, \end{aligned}$$

where we used the following obvious inequality $\frac{1}{\delta} w^{\frac{p}{2}+1} \geq \sigma w^{\frac{p}{2}}$. Thus we have from (4.4)

$$(4.6) \quad \begin{aligned} &\delta \int_\Omega G(\sigma(t))\xi^q(t) + \delta \iint_{(0,t) \times \Omega} w^{p/2-1} |\nabla \sigma|^2 \varphi(\sigma)\xi^q \\ &\leq \gamma \delta \iint_{(0,t) \times \Omega} G(\sigma)\xi^{q-1} \partial_\tau \xi + \gamma \delta \iint_{(0,t) \times \Omega} w^{\frac{p}{2}-1} |\nabla \xi|^2 \sigma^2 \varphi(\sigma)\xi^{q-2} + \frac{\gamma}{\delta} \iint_{(0,t) \times \Omega} F^2 w^{\frac{p}{2}+1} \varphi(\sigma)\xi^q. \end{aligned}$$

To complete the proof note that $\varphi(\sigma) \leq \frac{1}{\lambda}$, $G(\sigma) \leq \gamma \sigma^2$ and $w^{\frac{p}{2}+1} \leq \gamma(l^{p/2+1} + \delta^{p/2+1} \sigma^{p/2+1})$. \square

The next lemma provides the estimate of the last term in the right hand side of (4.3).

Lemma 4.2. *Let $h \in H_0^1(B) \cap L^\infty(B)$ be such that $-\Delta h = F^2$. Then*

$$\delta^{\frac{p}{2}-1} \iint F^2 \sigma^{\frac{p}{2}+1} \varphi(\sigma) \xi^q \leq \gamma \|h\|_\infty \iint |\nabla \sigma|^2 w^{\frac{p}{2}-1} \varphi(\sigma) \xi^q + \gamma \|h\|_\infty \iint |\nabla \xi|^2 \sigma^2 w^{\frac{p}{2}-1} \varphi(\sigma) \xi^{q-1}.$$

Proof. Recall that $\varphi(\sigma) \asymp \frac{\sigma}{\sigma+1}$ and apply (1.17). Then

$$\begin{aligned} \iint F^2 \sigma^{\frac{p}{2}+1} \varphi(\sigma) \xi^q &\leq \gamma \iint F^2 \frac{\sigma^{\frac{p}{2}+2}}{\sigma+1} \xi^q \\ &\leq \gamma \|h\|_\infty \iint |\nabla \sigma|^2 \sigma^{\frac{p}{2}-1} \frac{\sigma}{\sigma+1} \xi^q + \gamma \|h\|_\infty \iint |\nabla \xi|^2 \sigma^{\frac{p}{2}+1} \xi^{q-2}. \end{aligned}$$

Finally, note that $\delta^{\frac{p}{2}-1} \sigma^{\frac{p}{2}-1} \leq w^{\frac{p}{2}-1}$. □

Now let $(x_0, t_0) \in \Omega_T$. Given $r, \delta, l > 0$ we denote $\Delta := \max\{\delta, l\}$ and

$$Q = Q_{r,\Delta}^{x_0,t_0} = B_r(x_0) \times I_\Delta \equiv B_r(x_0) \times (t_0 - \frac{r^2}{\Delta^{\frac{p}{2}-1}}, t_0 + \frac{r^2}{\Delta^{\frac{p}{2}-1}}).$$

$\xi \in C_c^1(B_1(0) \times (-1, 1))$, $0 \leq \xi \leq 1$, $\xi = 1$ on $B_{\frac{1}{2}}(0) \times (-\frac{1}{2}, \frac{1}{2})$, $\xi_{r,\Delta}(x, t) := \xi(x_0 + r^{-1}x, t_0 + \Delta^{\frac{p}{2}-1}r^{-2}t)$. Then $\xi_{r,\Delta} \in C_c^1(Q_{r,\Delta}^{x_0,t_0})$, $0 \leq \xi_{r,\Delta} \leq 1$, $\xi_{r,\Delta} = 1$ on $\frac{1}{2}Q$, and $|\nabla \xi_{r,\Delta}| \leq \gamma r^{-1}$ and $|\partial_\tau \xi_{r,\Delta}| \leq \gamma \Delta^{\frac{p}{2}-1} r^{-2}$. Set

$$\Phi(w) = \int_0^{w^+} s^{\frac{p}{4}} (1+s)^{-\frac{1}{2}} ds \asymp \min\{w^{\frac{p}{4}+1}, w^{\frac{p+2}{4}}\}, \quad \Psi(w) = \int_0^{w^+} s^{\frac{1}{2}} (1+s)^{-\frac{1}{2}} ds \asymp \min\{w^{\frac{3}{2}}, w\}.$$

Corollary 4.3. *If $Q_{r,\Delta}^{x_0,t_0} \Subset \Omega_T$ then*

$$\begin{aligned} &\text{ess sup}_t \int G(\sigma(t)) \xi_{r,\Delta}^q(t) + \delta^{\frac{p}{2}-1} \iint |\nabla \Phi(\sigma)|^2 \xi_{r,\Delta}^q + l^{\frac{p}{2}-1} \iint |\nabla \Psi(\sigma)|^2 \xi_{r,\Delta}^q \\ &\leq \gamma \Delta^{\frac{p}{2}-1} r^{-2} \iint \sigma^2 \xi_{r,\Delta}^{q-2} + \gamma \delta^{\frac{p}{2}-1} r^{-2} \iint \sigma^{\frac{p}{2}+1} \xi_{r,\Delta}^{q-2} + \gamma \left(\frac{l}{\delta}\right)^2 r^2 \int_{B_r(x_0)} F^2. \end{aligned}$$

Set $r_j = r_0 2^{-j}$, $B_j = B_{r_j}(x_0)$, $l_{j+1} = l_j + \delta_j$, $l_0 = 1$, $\Delta_j = \max\{l_j, \delta_j\}$, $I_j = I_{\Delta_j}$, $Q_j = B_j \times I_j$, $\xi_j = \xi_{r_j, \Delta_j}$. With this notation the next lemma is easy to check.

Lemma 4.4. *If $\delta_j > (\frac{1}{2})^{\frac{2}{p-2}} \delta_{j-1}$ then $I_j \subset \frac{1}{2} I_{j-1}$.*

Let $L_j = \{(x, t) \in Q_j : w(x, t) > l_j\}$, $L_j(t) = \{x \in B_j : w(x, t) > l_j\}$. Fix $\varkappa > 0$ a small number which will be chosen later depending on the known data.

Define

$$\begin{aligned} A_j(l) &= \sup_{t \in I_j} \frac{1}{r_j^N} \int_{L_j(t)} G\left(\frac{w-l_j}{l-l_j}\right) \xi_j^q dx + \frac{(l-l_j)^{\frac{p}{2}-1}}{r_j^{N+2}} \iint_{L_j} \left(\frac{w-l_j}{l-l_j}\right)^{\frac{p}{2}+1} \xi_j^{q-2} dx dt \\ &\quad + \frac{\Delta_j(l)^{\frac{p}{2}-1}}{r_j^{N+2}} \iint_{L_j} \left(\frac{w-l_j}{l-l_j}\right)^2 \xi_j^{q-2} dx dt, \end{aligned} \tag{4.7}$$

where $\Delta_j(l) = \max\{l_j, l-l_j\}$.

Set

$$F_j = \left(\frac{1}{r_j^{N-2}} \int_{B_j} F^2(x) dx \right)^{\frac{1}{2}}, \quad j = 1, 2, \dots$$

The sequence $(l_j)_{j \in \mathbb{N}}$ is defined inductively. We set as above $l_0 = 1$. Suppose l_1, \dots, l_j have been defined. We show how to define l_{j+1} .

First, note that $A_j(l)$ is continuous and $A_j(l) \rightarrow 0$ as $l \rightarrow \infty$. If $A_j(l_j + F_j) \leq \varkappa$ then we set $l_{j+1} = l_j + F_j$. If on the other hand $A_j(l_j + F_j) > \varkappa$ then there exists $\tilde{l} > l_j + F_j$ such that $A_j(\tilde{l}) = \varkappa$, and we set $l_{j+1} = \tilde{l}$. In both cases

$$(4.8) \quad A_j(l_{j+1}) \leq \varkappa.$$

Lemma 4.5.

$$(4.9) \quad \delta_j \leq \left(\frac{1}{2}\right)^{\frac{2}{p-2}} \delta_{j-1} + \gamma l_j F_j.$$

Proof. Fix $j \geq 1$ and suppose that $\delta_j > \left(\frac{1}{2}\right)^{\frac{2}{p-2}} \delta_{j-1}$ and $\delta_j > F_j$ since otherwise there is nothing to prove. This implies that $A_j(l_{j+1}) = \varkappa$.

We denote $\sigma_j := \frac{w-l_j}{\delta_j}$, $\Phi_j := \Phi(\sigma_j)$, $\Psi_j := \Psi(\sigma_j)$.

Claim. $\sup_{t \in I_j} \frac{1}{r_j^N} |L_j(t)| \leq \gamma \varkappa$. Indeed, for $(x, t) \in L_j$ one has

$$(4.10) \quad \frac{w(x, t) - l_{j-1}}{\delta_{j-1}} = 1 + \frac{w(x, t) - l_j}{\delta_{j-1}} \geq 1.$$

Note that Lemma 4.4 yields $\xi_{j-1} = 1$ on Q_j . Hence

$$\begin{aligned} r_j^{-N} \sup_{t \in I_j} |L_j(t)| &\leq r_j^{-N} \sup_{t \in I_j} \int_{L_j(t)} G\left(\frac{w-l_{j-1}}{\delta_{j-1}}\right) \xi_{j-1}^q dx \\ &\leq 2^N r_{j-1}^{-N} \sup_{t \in I_{j-1}} \int_{L_{j-1}(t)} G\left(\frac{w-l_{j-1}}{\delta_{j-1}}\right) \xi_{j-1}^q dx \leq 2^N \varkappa, \end{aligned}$$

which proves the claim.

Now decompose L_j as $L_j = L'_j \cup L''_j$,

$$(4.11) \quad L'_j = \left\{ (x, t) \in L_j : \frac{w(x, t) - l_j}{\delta_j} < \varepsilon \right\}, \quad L''_j = L_j \setminus L'_j,$$

where ε depending on the data is small enough to be determined later. Then

$$(4.12) \quad \frac{\delta_j^{\frac{p}{2}-1}}{r_j^{N+2}} \iint_{L'_j} \sigma_j^{\frac{p}{2}+1} \xi_j^{q-2} dx dt + \frac{\Delta_j^{\frac{p}{2}-1}}{r_j^{N+2}} \iint_{L''_j} \sigma_j^2 \xi_j^{q-2} dx dt \leq \gamma \varepsilon^2 (1 + \varepsilon^{p/2-1}) \sup_{t \in I_j} \frac{1}{r_j^N} |L_j(t)| \leq \gamma \varepsilon^2 \varkappa.$$

Now recall that $\Phi(\sigma) \asymp \min\{\sigma^{\frac{p}{4}+1}, \sigma^{\frac{p+2}{4}}\}$. So $\sigma_j^{\frac{p}{2}+1} \leq \gamma(\varepsilon) \Phi_j^2$ on L''_j . So we have

$$\begin{aligned} (4.13) \quad &\frac{\delta_j^{\frac{p}{2}-1}}{r_j^{N+2}} \iint_{L''_j} \sigma_j^{\frac{p}{2}+1} \xi_j^{q-2} dx dt \leq \gamma(\varepsilon) \frac{\delta_j^{\frac{p}{2}-1}}{r_j^{N+2}} \iint_{L''_j} \left(\Phi_j \xi_j^{\frac{q}{2}-1}\right)^2 dx dt \\ &\leq \gamma(\varepsilon) \frac{\delta_j^{\frac{p}{2}-1}}{r_j^{N+2}} \int_{I_j} |L_j(t)|^{\frac{2}{N}} \left(\int_{L_j(t)} \left(\Phi_j \xi_j^{\frac{q}{2}-1}\right)^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} dt \\ &\leq \gamma(\varepsilon) \frac{\delta_j^{\frac{p}{2}-1}}{r_j^{N+2}} \left(\sup_{t \in I_j} |L_j(t)| \right)^{\frac{2}{N}} \iint_{L_j} \left| \nabla (\Phi_j \xi_j^{\frac{q}{2}-1}) \right|^2 dx dt \\ &\leq \gamma(\varepsilon) \left(\sup_{t \in I_j} \frac{1}{r_j^N} |L_j(t)| \right)^{\frac{2}{N}} \frac{\delta_j^{\frac{p}{2}-1}}{r_j^N} \iint_{L_j} \left| \nabla (\Phi_j \xi_j^{\frac{q}{2}-1}) \right|^2 dx dt \\ &\leq \gamma(\varepsilon) \varkappa^{\frac{2}{N}} \frac{\delta_j^{\frac{p}{2}-1}}{r_j^N} \iint_{L_j} \left| \nabla (\Phi_j \xi_j^{\frac{q}{2}-1}) \right|^2 dx dt. \end{aligned}$$

Similarly, if $l_j \geq \delta_j$,

$$(4.14) \quad \begin{aligned} \frac{\Delta_j^{\frac{p}{2}-1}}{r_j^{N+2}} \iint_{L_j''} \sigma_j^2 \xi_j^{q-2} dx dt &\leq \gamma(\varepsilon) \left(\sup_{t \in I_j} \frac{1}{r_j^N} |L_j(t)| \right)^{\frac{2}{N}} \frac{l_j^{\frac{p}{2}-1}}{r_j^N} \iint \left| \nabla(\Psi_j \xi_j^{\frac{q}{2}-1}) \right|^2 dx dt \\ &\leq \gamma(\varepsilon) \varkappa^{\frac{2}{N}} \frac{l_j^{\frac{p}{2}-1}}{r_j^N} \iint \left| \nabla(\Psi_j \xi_j^{\frac{q}{2}-1}) \right|^2 dx dt. \end{aligned}$$

Using Corollary 4.3 we have

$$(4.15) \quad \begin{aligned} \frac{\delta_j^{\frac{p}{2}-1}}{r_j^{N+2}} \iint_{L_j''} \sigma_j^{\frac{p}{2}+1} \xi_j^{q-2} dx dt + \frac{\Delta_j^{\frac{p}{2}-1}}{r_j^{N+2}} \iint_{L_j''} \sigma_j^2 \xi_j^{q-2} dx dt \\ \leq \gamma(\varepsilon) \varkappa^{\frac{2}{N}} \left[\varkappa + \left(\frac{l_j}{\delta_j} \right)^2 r_j^{2-N} \int_{B_j} F^2 dx \right]. \end{aligned}$$

Now we estimate the first term in the right hand side of (4.7) using Corollary (4.3) and the Claim.

$$(4.16) \quad \begin{aligned} \sup_{t \in I_j} \frac{1}{r_j^N} \int_{L_j(t)} G \left(\frac{w - l_j}{l - l_j} \right) \xi_j^q dx \\ \leq \gamma \varepsilon^2 (1 + \varepsilon^{p/2-1}) \varkappa + \left(\frac{l_j}{\delta_j} \right)^2 F_j^2 + \gamma(\varepsilon) \varkappa^{\frac{2}{N}} \left[\varkappa + \left(\frac{l_j}{\delta_j} \right)^2 r_j^{2-N} \int_{B_j} F^2 dx \right]. \end{aligned}$$

Collecting (4.12)–(4.16) we obtain

$$\varkappa \leq \gamma \varepsilon^2 (1 + \varepsilon^{p/2-1}) \varkappa + \left(\frac{l_j}{\delta_j} \right)^2 F_j^2 + \gamma(\varepsilon) \varkappa^{\frac{2}{N}} \left[\varkappa + \left(\frac{l_j}{\delta_j} \right)^2 F_j^2 \right].$$

Now first choosing ε by the condition

$$\gamma \varepsilon^2 (1 + \varepsilon^{p/2-1}) = \frac{1}{4},$$

and then \varkappa such that

$$\gamma(\varepsilon) \varkappa^{\frac{2}{N}} = \frac{1}{4},$$

we arrive at (4.9). □

Summing up the inequalities (4.9) with respect to j from 1 to $J-1$ we obtain

$$l_J \leq \gamma \delta_0 + \gamma l_J \sum_{j=1}^{J-1} F_j.$$

Choosing r_0 small enough so that $\int_0^{r_0} \frac{dr}{r} \left(\frac{1}{r^{N-2}} \int_{B_r(x_0)} F^2(y) dy \right)^{\frac{1}{2}} < \frac{1}{2\gamma}$ we arrive at

$$(4.17) \quad l_J \leq \gamma \delta_0.$$

It remains to estimate δ_0 . From (4.7) we have

$$\delta_0 \leq \left(\frac{1}{r_0^N} \sup_t \int_{B_0} |\nabla u|^4 \xi_0^q dx \right)^{\frac{1}{2}} + \left(\frac{1}{r_0^{N+p}} \iint_{Q_0} |\nabla u|^{p+2} \xi_0^{q-2} dx d\tau \right)^{\frac{1}{2}}.$$

By the iteration argument of Proposition 2.3 with $\alpha = l = 1$, we obtain, with $Q_0 \Subset Q \Subset \Omega_T$,

$$(4.18) \quad \delta_0 \leq \gamma(r_0^{-\frac{N}{2}} + r_0^{-\frac{N+p}{2}}) \left(\iint_Q |\nabla u|^p dx dt + \iint_Q (F^2 + 1) dx dt + \left(\iint_Q (F^2 + 1) dx dt \right)^{\frac{N}{2(N+2)}} \right).$$

It follows from (4.17) that the sequence $(l_j)_j$ converges to a limit $l \leq \gamma\delta_0$, and $\delta_j \rightarrow 0$ as $j \rightarrow \infty$. We conclude from (4.8) that

$$\frac{1}{r_j^{N+2}} \iint_{B_j \times (t_0 - \frac{r_j^2}{lp/2-1}, t_0 + \frac{r_j^2}{lp/2-1})} (|\nabla u(x, t)|^2 - l)_+^{p/2+1} dx dt \leq \gamma\delta_j^2 \rightarrow 0 \quad (j \rightarrow \infty).$$

Choosing (x_0, t_0) as a Lebesgue point of the function $(|\nabla u|^2 - l)_+^{p/2+1}$ we conclude that $|\nabla u(x_0, t_0)| \leq l^{1/2} \leq \gamma\delta_0^{1/2}$ with δ_0 estimated in (4.18).

A Appendix: Example

Here we construct a function $f \in L^1(\mathbb{R}^N)$ with compact support such that $\sup_x W_p^f(x, \infty) < \infty$ however $\lim_{R \rightarrow 0} \sup_x W_p^f(x, R) > 0$. It is a generalization of an example in the celebrated paper by Aizenman and Simon [2, Appendix 1, Example 1].

Example A.1. Let $p \in [2, N)$. Fix a sequence $\{\rho_n\} \subset (0, 1)$ such that $\rho_n \downarrow 0$ as $n \rightarrow \infty$ and $\sum_n \rho_n^{\frac{N-p}{N-1}} < \infty$, and a bounded sequence $\{x_n\} \subset \mathbb{R}^N$ such that $|x_n - x_m| \geq 4\rho_n^{\frac{N-p}{N-1}}$ for $m \neq n$. Let $f_n := \rho_n^{-p} \mathbf{1}_{B_{\rho_n}(x_n)}$ and $f = \sum_n f_n$. Let ω_N denote the volume of the unit ball in \mathbb{R}^N . First, note that

$$(A.1) \quad W_p^{f_n}(x_n, \rho_n) = \int_0^{\rho_n} \frac{dr}{r} \left(r^p \rho_n^{-p} \omega_N \right)^{\frac{1}{p-1}} = \frac{p-1}{p} \omega_N^{\frac{1}{p-1}} =: a_p.$$

Next, let $|x - x_n| < \rho_n + \rho_n^{\frac{N-p}{N-1}}$. Then

$$(A.2) \quad W_p^{f_n}(x, \infty) \leq W_p^{f_n}(x_n, \infty) = W_p^{f_n}(x_n, \rho_n) + \int_{\rho_n}^{\infty} \frac{dr}{r} \left(r^{p-N} \rho_n^{N-p} \omega_N \right)^{\frac{1}{p-1}} = a_p + \frac{p-1}{N-p} \omega_N^{\frac{1}{p-1}} =: b_p.$$

Now let $|x - x_n| \geq \rho_n + \rho_n^{\frac{N-p}{N-1}}$. Then

$$(A.3) \quad \begin{aligned} W_p^{f_n}(x, \infty) &= \int_{|x-x_n|-\rho_n}^{\infty} \frac{dr}{r} \left(r^{p-N} \rho_n^{-p} |B_r(x) \cap B_{\rho_n}(x_n)| \right)^{\frac{1}{p-1}} \\ &\leq \omega_N^{\frac{1}{p-1}} \int_{\rho_n^{\frac{N-p}{N-1}}}^{\infty} \frac{dr}{r} \left(\frac{r}{\rho_n} \right)^{\frac{p-N}{p-1}} = \frac{p-1}{N-p} \omega_N^{\frac{1}{p-1}} \rho_n^{\frac{N-p}{N-1}} =: c_p \rho_n^{\frac{N-p}{N-1}}. \end{aligned}$$

Observe that if $|x - x_n| < \rho_n + \rho_n^{\frac{N-p}{N-1}}$ for some $n \in \mathbb{N}$ then $|x - x_m| > \rho_m + \rho_m^{\frac{N-p}{N-1}}$ for every $m \neq n$. Since $p \geq 2$, it follows from (A.2) and (A.3) that

$$W_p^f(x, \infty) \leq \sum_n W_p^{f_n}(x, \infty) \leq b_p + c_p \sum_n \rho_n^{\frac{N-p}{N-1}} < \infty.$$

On the other hand, (A.1) implies that

$$\lim_{R \rightarrow 0} \sup_{x \in \mathbb{R}^N} W_p^f(x, R) \geq \lim_{n \rightarrow \infty} W_p^{f_n}(x_n, \rho_n) = a_p > 0.$$

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